Calculus of Variations

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Preface

The present work is the result of the traineeship program of the *Mathematics & Applied Mathematics Department* of the *University of Crete* the writer has participated during the summer of the year 2015. The traineeship was conducted under the carrier *Foundation for Research & Technology* (FORTH) which assigned as supervisor of the project the professor of the *Mathematics & Applied Mathematics Department*, Achilles K. Tertikas, who is a collaborator of the carrier working under the *Institute of Applied & Computational Mathematics*.

The purpose of the traineeship was the study of the book *Introduction to the Calculus of Variations* by Bernard Dacorogna (published by *Imperial College Press* in 2004) and in turn the composition of a work with the same subject. Based on the above book, this piece of writing follows the same structure and "storyline". It aims to provide a detailed presentation of the subject and hopes to help the reader understand the essence of the theory as well as to elucidate and eventually surmount the technical difficulties one may encounter. The writer hopes that the lack of illustrative examples will not be a hindrance to the study and the understanding of this delicate theory. For extra exercises and for further analysis and applications of the theory presented here one should refer to the book of B. Dacorogna.

I have to thank professor Tertikas for all the time he spent listening to my thoughts and answering my questions. I believe his efforts prevented a great deal of misunderstandings that I would otherwise have made.

However, being the work of a human, this essay is bound to have omissions, vagueness and mistakes. I apologize for any inaccurate or false results for which I shall bare full responsibility.

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Chapter 0

Introduction

Brief Historical Notes

Calculus of Variations is one the classical branches of Mathematics. However modern its name may sound, the truth is that it is a rather old subject originating back at ancient Greece and, in particular, it is closely related to the isoperimetric problem. This field of mathematics it is still active nowadays and it has passed through many famous names such as Archimedes, Zenodorus, Pappus, - at its early ages and later through - Euler, Galileo, Lagrange, Legendre, Simpson, Carathéodory, Frobenius, Dirichlet, Hurwitz, Hilbert, Weierstrass, Lebesgue, Minkowski, H.A.Schwarz, Tonelli and Sobolev to name but a few.

A station at history of Calculus of Variations was the problem of the brachistochrone formulated by Galileo at 1685, which states: If we let a point-particle with mass slide from a point A to a point B under only the force of gravity, what is the route it must follow in order to reach B at the least amount of time? John Bernoulli, and later other mathematicians, found that the trajectory has to be a *cycloid*, but it was not until Euler and Lagrange, who introduced the *Euler-Lagrange equations*, that we had a systematic approach for this kind of problems. The connection of this field with (partial) differential equations had since early become evident.

Another important problem that mathematicians encountered was the so-called Dirichlet integral, which happened to have a strong correlation with the Laplace equation. The attempts to deal with the Dirichlet integral invited new methods into the scene that would be called *direct methods* on the contrary to the previous methods known as *classical methods*. However, there were still some key elements that had yet to be found, despite all the hard efforts.

Eventually, at the start of the 20th century the contribution of Lebesgue, Sobolev and others drived to the establishment of what is known today as *Sobolev spaces*. These spaces were formulated using the notion of the *weak derivative* and proved to be the right framework under which one could expect for solutions of certain minimizing problems to exist. Nevertheless, problems regarding the regularity of such solutions had to be addressed. Thankfully, the techniques of the direct methods provided enough tools to confront this difficulties under some extra hypotheses.

These developments gave a new boost to the field and many important results appeared thereafter. The applications of the theory had been further extended and soon the mathematical society could deal quite efficiently with problems such as the *Plateau problem*, a variant of the problem of minimal surfaces, and the isoperimetric inequality.

History with Formulas

The general problem that this theory attempts to answer is whether there is a minimizer of the problem

$$\inf \left\{ I(u) = \int_a^b f\left(x, u(x), u'(x)\right) \, dx \ : \ u \in X \right\} = m \tag{P}$$

when $f : (a, b) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and X is the set of all admissible functions with a prescribed boundary, say $u(a) = \alpha$ and $u(b) = \beta$, and how regular such a minimizer is if it exists.

Starting not long after the birth of *Infinitesimal Calculus* by Newton and Leibniz, the so-called classical methods were developed to deal with the problem. It is common practice nowadays to study the equation $F'(x) = 0$ in order to find the stationary points a function and therefore studying higher derivatives to determine whether these points are global or local minimizers or just saddle points of the function. The *Euler-Lagrange equation*

$$\frac{d}{dx} f_\xi \left(x, u(x), u'(x)\right) = f_u \left(x, u(x), u'(x)\right)$$

is the equivalent to $F'(x) = 0$ of this "classical" approach. Nevertheless, this equation itself seems to be quite restrictive, because the function f is not necessarily differentiable.

It is natural, then, for one to wonder if this equation has always a solution and generally if the problem itself has always a solution. At the time, it was widely believed that all such integrals considered in the set of all continuously differentiable functions admitted a minimizer and this was known as *the Dirichlet principle*. Unfortunately, this is not the case as Weierstrass showed with a surprisingly simple counterexample: If we consider the integrand $f(x, u(x)) = x^2 (u'(x))^2$ in the space $X = \{u \in C^1([-1, 1]) \ : \ u(-1) = -1 \textit{ and } u(1) = 1\}$ of all the continuously differentiable functions, then it is clear that

$$I(u) = \int_{-1}^{1} x^2 (u'(x))^2 \, dx \geq 0$$

At the same time, the sequence (u_n) with

$$u_n(x) = \frac{\arctan \frac{x}{n}}{\arctan \frac{1}{n}}, \qquad \forall x \in [-1, 1]$$

lives in X and $I(u_n) \to 0$ while $n \to +\infty$. So, the infimum of I has to be 0. This implies that the minimizer $\overline{u}$ of I has to be constant, but there are no constant functions in X. Hence, in this example I fails to have a minimizer.

This led mathematicians to invent new techniques and tools to address the problem. The main idea was to split the question in two separate problems: the existence of a minimizer and the regularity of the minimizer if it existed. For the existence, it was needed to have a sequence (u_n) that would converge to a minimizer $\overline{u} \in X$. The usual notion of convergence $u_n \to \overline{u}$, however, proved to be rather insufficient for the purpose for it was way too strong. So, it had to be weakened to a more "relaxed" form and thus *weak convergence* $u_n \rightharpoonup \overline{u}$ came about. This allowed for sequences to be *compact* and if we can additionally establish the lower semi-continuity of I, that is if $\liminf_{n \to +\infty} I(u_n) \geq I(\overline{u})$, then the existence of a minimizer is guaranteed.

The regularity of the minimizer is closely related to the regularity of the function f. In fact, under some conditions it is possible to obtain that $\overline{u} \in C^k$ whenever $f \in C^k$. The set of techniques used to deal with these problems were developed by Hilbert, Lebesgue and Tonelli and are known as *direct methods* contrary to the "old" classical methods. Their work illuminated many elements that were unknown till then and constructed a more firm basis of the theory.

However, the question remained: What is the most general framework we can work on under which we can hope to get a minimizer? It was not until the middle of the 20th when the notion of the *weak derivative* and *Sobolev spaces*, consisting of weakly differentiable functions, came into the scene. In particular, weakly differentiable functions consist of a wider diversity of functions than just the differentiable ones and allow one to integrate by parts when the functions participating need not be differentiable, but only weakly differentiable. Their definition uses some *test functions*, which have to be relatively regular, and is based on the fact that integrals do not "feel" sets of measure zero. Sobolev spaces, denoted with $W^{k,p}$, are then nothing more than collection of such functions and seem to be the suitable spaces we so hard have searched for.

In this piece of writing we initially present the notion of weak derivatives and some important results regarding Sobolev spaces. Then, adhering to the "historical structure" of the theory, we study the classical methods that showed up through the years. Immediately thereafter, the direct methods follow. Firstly, we try to establish the existence of minimizers and later we seek to determine how regular these minimizers are. But always under the perspective of Sobolev spaces.

Contents

Chapter 1

Preliminaries

1.1 Introduction

In this introductory chapter one should find the tools and the theorems that will be used throughout our study. We begin with the definitions of the continuously differentiable functions along with the Hölder continuous functions and after that we see Lebesgue's L^p spaces. We cannot state all the classical theorems of Measure Theory, but we will see some famous results such as the Hölder inequality and Riesz's theorem. Next we introduce Sobolev spaces that popped into existence through the study our theory as the most appropriate framework for minimizers of our problems to exist. Last but not least, we present some convexity results that are particularly useful in the classical approach of our problems.

1.2 Basic Definitions

Definition 1.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Then we define the following:*

i) $C^0(\Omega) = C(\Omega)$ *is the* set of continuous functions $u : \Omega \to \mathbb{R}$ *and* $C^0(\Omega; \mathbb{R}^m) = C(\Omega; \mathbb{R}^m)$ *is the* set of continuous maps $u : \Omega \to \mathbb{R}^m$.

ii) $C^0(\overline{\Omega}) = C(\overline{\Omega})$ *is the set of continuous functions $u : \Omega \to \mathbb{R}$ which can be continuously extended to $\overline{\Omega}$ and similarly with the continuous maps $C^0(\overline{\Omega}; \mathbb{R}^m) = C(\overline{\Omega}; \mathbb{R}^m)$.*

iii) The support *of a function $u : \Omega \to \mathbb{R}$ is as usual defined as* $\operatorname{supp} u = \overline{\{x \in \Omega \ : \ u(x) \neq 0\}}$.

iv) $C_0(\Omega) = \{u \in C(\Omega) \ : \ \operatorname{supp} u \subset \Omega \text{ is compact}\}$.

v) The norm over $C(\overline{\Omega})$ is defined by $\|u\|_{C^0} = \sup\limits_{x \in \overline{\Omega}} \{|u(x)|\}$ *and turns $C(\overline{\Omega})$ into a* Banach space.

Theorem 1.2 (Ascoli-Arzela Theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open and connected set, i.e. a bounded domain, and $\mathcal{F} \subset C(\overline{\Omega})$ be a set of uniformly bounded functions. If the following property of equicontinuity holds*

$$\forall \varepsilon > 0 \text{ there exists } \delta > 0 \text{ so that}$$
$$|x - y| < \delta \implies |u(x) - u(y)| < \varepsilon, \qquad \forall x, y \in \overline{\Omega}, \ \forall u \in \mathcal{F}$$

then $\overline{\mathcal{F}}$ is compact.

Throughout this text we will be using the usual notation for the derivative and for the partial derivatives. However, we have to present here the symbols for the so-called higher derivatives. Consider an integer $k \geq 1$. Then, the set

$$\mathcal{A}_k^n = \mathcal{A}_k = \left\{ a = (a_1, \ldots, a_n) \ : \ a_i \in \mathbb{N} \ \wedge \ \sum_{i=1}^{n} a_i = k \right\}$$

is the set of all multi-indices of k. For the elements of $\mathcal{A}$ we will sometimes also write $|a| = \sum_{i=1}^{n} a_i = k$. If $a \in \mathcal{A}$, then we define

$$D^a u = D_1^{a_1} \ldots D_n^{a_n} u = \frac{\partial^{|a|} u}{\partial x_1^{a_1} \ldots \partial x_n^{a_n}}$$

Definition 1.3. *Let $\Omega \subset \mathbb{R}^n$ be open and $k \geq 0$ be an integer. Then,*

i) $C^k(\Omega)$ *is the set of the functions $u : \Omega \to \mathbb{R}$ with all their partial derivatives continuous, that is if $u \in C^k(\Omega)$, then $D^a u$ with $a \in \mathcal{A}_l$ is continuous for every integer $l \in [0, k]$.*

ii) $C^k(\overline{\Omega})$ *is the set of the $C^k(\Omega)$ functions whose derivatives (up to order k) can be extended continuously to $\overline{\Omega}$. This space is a Banach space equipped with the following norm*

$$\|u\|_{C^k(\overline{\Omega})} = \max_{0 \leq |a| \leq k} \left\{ \sup_{x \in \overline{\Omega}} \{ |D^a u(x)| \} \right\}$$

iii) $C_0^k(\Omega) = C^k(\Omega) \cap C_0(\Omega)$ *is the space of $C^k(\Omega)$ functions that happen to be 0 outside some compact subset of Ω, i.e. they have compact support.*

iv) $C^\infty(\Omega) = \bigcap_{k=0}^{\infty} C^k(\Omega)$ *and* $C^\infty(\overline{\Omega}) = \bigcap_{k=0}^{\infty} C^k(\overline{\Omega})$

v) $C_0^\infty(\Omega) = C^\infty(\Omega) \cap C_0(\Omega)$

When u is a map from $\Omega \subset \mathbb{R}^n$ to $\mathbb{R}^m$ we define the respective sets of functions similarly.

Suppose we have a set $D \subset \mathbb{R}^n$ and a number $0 < \alpha \leq 1$. For a function $u : D \to \mathbb{R}$ we set

$$[u]_{C^{0,\alpha}(D)} = \sup_{\substack{x,y \in D \\ x \neq y}} \left\{ \frac{|u(x) - u(y)|}{|x - y|^\alpha} \right\}$$

We have, then, the following useful definitions.

Definition 1.4. *Let $\Omega \subset \mathbb{R}^n$ be open and $k \geq 0$ be an integer.*

i) We will denote as $C^{0,\alpha}(\Omega)$ the set of all $C(\Omega)$ functions, u, that satisfy

$$[u]_{C^{0,\alpha}(K)} = \sup_{\substack{x,y \in K \\ x \neq y}} \left\{ \frac{|u(x) - u(y)|}{|x-y|^\alpha} \right\} < +\infty$$

*for every compact set $K \subset \Omega$. This space is called the space of α-*Hölder continuous functions *or simply the space of* Hölder continuous functions*.*

ii) $C^{0,\alpha}(\overline{\Omega})$ is the set of all $C(\overline{\Omega})$ functions with

$$[u]_{C^{0,\alpha}(\overline{\Omega})} < +\infty$$

This space is a Banach space equipped with the norm

$$\|u\|_{C^{0,\alpha}(\overline{\Omega})} = \|u\|_{C^0(\overline{\Omega})} + [u]_{C^{0,\alpha}(\overline{\Omega})}$$

iii) $C^{k,\alpha}(\Omega)$ is the set of all $C^k(\Omega)$ functions with

$$[D^a u]_{C^{0,\alpha}(K)} = \sup_{\substack{x,y \in K \\ x \neq y}} \left\{ \frac{|D^a u(x) - D^a u(y)|}{|x-y|^\alpha} \right\} < +\infty$$

for every compact set $K \subset \Omega$ and for every multi-index $a \in \mathcal{A}_k$.

iv) $C^{k,\alpha}(\overline{\Omega})$ is the set of all $C^k(\overline{\Omega})$ functions such that

$$[D^a u]_{C^{k,\alpha}(\overline{\Omega})} < +\infty$$

for every multi-index $a \in \mathcal{A}_k$. And again the norm

$$\|u\|_{C^{k,\alpha}(\overline{\Omega})} = \|u\|_{C^k(\overline{\Omega})} + \max_{a \in \mathcal{A}_k} \left\{ [D^a u]_{C^{0,\alpha}(\overline{\Omega})} \right\}$$

turns $C^{k,\alpha}(\overline{\Omega})$ into a Banach space.

Remark. *By abuse of notation, for $\alpha = 0$ we write $C^{k,0}(\Omega) = C^k(\Omega)$. Also note that for $\alpha = 1$ $C^{0,1}(\overline{\Omega})$ is the space of all* Lipschitz continuous functions*, that is the functions for which there exists a constant $L \geq 0$ so that*

$$|u(x) - u(y)| \leq L|x-y|, \qquad \forall x, y \in \overline{\Omega}$$

The best such constant is by definition $L = [u]_{C^{0,1}(\overline{\Omega})}$.

And here is our first proposition regarding all those Hölder spaces.

Proposition 1.5. *Let $\Omega \subset \mathbb{R}^n$ be open and $\alpha \in [0,1]$. Then, the following hold:*

i) $u, v \in C^{0,\alpha}(\overline{\Omega}) \implies uv \in C^{0,\alpha}(\overline{\Omega})$

ii) Let $k \geq 0$ be an integer and $0 \leq \alpha \leq \beta \leq 1$. Then,

$$C^{k,1}(\overline{\Omega}) \subset C^{k,\beta}(\overline{\Omega}) \subset C^{k,\alpha}(\overline{\Omega}) \subset C^k(\overline{\Omega})$$

iii) If Ω is additionally bounded and convex, then

$$C^{k+1}(\overline{\Omega}) \subset C^{k,1}(\overline{\Omega})$$

In the rest of the text when there is no danger of confusion we will sometimes omit the domain of the functions from our notation. For example, we will only write C^k instead of $C^k(\Omega)$.

1.3 L^p spaces

Definition 1.6. *Let $\Omega \subset \mathbb{R}^n$ be open.*

i) For every $1 \leq p < +\infty$ the set of all measurable functions $u : \Omega \to \mathbb{R}$ with

$$\int_\Omega |u(x)|^p \, dx < +\infty$$

will be denoted with $L^p(\Omega)$.

ii) The set of all measurable functions $u : \Omega \to \mathbb{R}$ that are bounded almost everywhere in Ω will be denoted with $L^\infty(\Omega)$.

The above spaces are Banach spaces when equipped with the norm

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p \, dx \right)^{\frac{1}{p}}, \quad for\ 1 \leq p < +\infty$$

$$\|u\|_{L^\infty(\Omega)} = \inf\{M \ : \ |u(x)| \leq M \ almost\ everywhere\ in\ \Omega\}, \quad for\ p = +\infty$$

We can, then, define similarly the respective spaces for maps $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ where $u = (u^1, u^2, \dots, u^m)$ demanding $u^i \in L^p(\Omega)$ for every $i = 1, 2, \dots, m$ and write $u \in L^p(\Omega; \mathbb{R}^m)$.

Remark. *The quantity $\|u\|_{L^\infty}$ is sometimes called the* essential supremum *of u in Ω and is denoted with* $\operatorname*{esssup}_{x\in\Omega}\{|u(x)|\}$.

Theorem 1.7. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $1 \leq p \leq +\infty$.*

i) The space $L^2(\Omega)$, in addition to being a Banach space, is a Hilbert space with inner product

$$\langle u\,; v \rangle = \int_\Omega u(x)v(x)\, dx$$

ii) **Hölder inequality.** *For every $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$, the function uv lives in $L^1(\Omega)$ and the following inequality holds:*

$$\|uv\|_{L^1} \leq \|u\|_{L^p} \|v\|_{L^q}$$

Observe that the case when $p = q = 2$ is exactly the famous Cauchy-Schwartz inequality

$$\int_\Omega |u(x)v(x)|\, dx \leq \left(\int_\Omega |u(x)|^2 \, dx \right)^{\frac{1}{2}} \left(\int_\Omega |v(x)|^2 \, dx \right)^{\frac{1}{2}}$$

iii) **Minkowski inequality.** *For $u, v \in L^p(\Omega)$ is holds that*

$$\|u + v\|_{L^p} \leq \|u\|_{L^p} + \|v\|_{L^p}$$

iv) **Riesz Theorem.** *Let $(L^p)^\star$ be the dual space of L^p. If $\frac{1}{p}+\frac{1}{q}=1$ with $1 \le p < +\infty$, then $(L^p)^\star$ can be identified with L^q. This means that if $\phi \in (L^p)^\star$, then there exists a unique $u \in L^q$ such that*

$$\langle \phi\,;f\rangle = \phi(f) = \int_\Omega u(x)f(x)\,dx, \qquad \forall f \in L^p(\Omega)$$

as well as

$$\|u\|_{L^q} = \|\phi\|_{(L^p)^\star}$$

Pay close attention to the fact that the theorem is false when $p=+\infty$, that is the dual space of L^∞ is not actually L^1. In fact, we have that $L^1 \subsetneq (L^\infty)^\star$ whereas $(L^1)^\star \cong L^\infty$.

v) *For $1 \le p < +\infty$ the space L^p is separable and for $1 < p < +\infty$ it is reflexive, which means that its bidual space (the dual of its dual) is the same space, i.e. $(L^p)^{\star\star} \cong L^p$.*

vi) *For $1 \le p < +\infty$ the piecewise constant functions (also known as step functions) are dense in L^p. This means that for every $u \in L^p$ there exists a sequence (u_n) of piecewise constant functions such that*

$$\lim_{n\to+\infty} \|u_n - u\|_{L^p} = 0$$

Exactly the same holds for the C_0^∞ functions. Note that the above are false in the case $p=+\infty$.

Throughout the text we will not make any distinction between the spaces $(L^p)^\star$ and L^q and will simply write $(L^p)^\star = L^q$ for every $1 < p < +\infty$ where $\frac{1}{p}+\frac{1}{q}=1$.

Its about time we see now the notion of convergence in these L^p spaces. The usual and natural notion will be called here *strong* whereas we will define another convergence known as *weak*. In particular, we give the following definitions:

Definition 1.8. *Let $\Omega \subset \mathbb{R}^n$ be open.*

i) *If $1 \le p \le +\infty$, we will say that a sequence (u_n)* strongly converges to u in $L^p(\Omega)$ *when $u_n, u \in L^p$ and*

$$\lim_{n\to+\infty} \|u_n - u\|_{L^p} = 0$$

We will write, then, "$u_n \to u$ in L^p" (or sometimes $u_n \overset{L^p}{\to} u$).

ii) *If $1 \le p < +\infty$, we will say that a sequence (u_n)* weakly converges to u in $L^p(\Omega)$ *when $u_n, u \in L^p$ and*

$$\lim_{n\to+\infty} \int_\Omega (u_n(x)-u(x))\varphi(x)\,dx = 0, \qquad \forall \varphi \in L^q(\Omega)$$

where $\frac{1}{p}+\frac{1}{q}=1$. We will write, then, "$u_n \rightharpoonup u$ in L^p" (or sometimes $u_n \overset{L^p}{\rightharpoonup} u$).

iii) *If $p=+\infty$, we will say that a sequence (u_n)* weakly $*$ converges in $L^\infty(\Omega)$ *when $u_n, u \in L^\infty$ and*

$$\lim_{n\to+\infty} \int_\Omega (u_n(x)-u(x))\varphi(x)\,dx = 0, \qquad \forall \varphi \in L^1(\Omega)$$

We will write, then, "$u_n \overset{*}{\rightharpoonup} u$ in L^∞".

Remark. *i) The reason why we separated the two cases of weak and weak $*$ convergence, even though they have the same form, is because the space L^1 is not the whole dual space of L^∞. Nevertheless, the so-called* test functions, *φ, live only in L^1.*

ii) Every limit described above is unique.

iii) One can see that strong convergence is stronger than weak convergence (obviously!). This means that

$$u_n \to u \text{ in } L^p \implies \begin{cases} u_n \rightharpoonup u \text{ in } L^p & \text{for } 1 \le p < +\infty \\ u_n \overset{*}{\rightharpoonup} u \text{ in } L^\infty & \text{for } p = +\infty \end{cases}$$

Proposition 1.9. *If $\Omega \subset \mathbb{R}^n$ is a bounded open set, then the following properties hold:*

i) For every $1 \le p < +\infty$, if $u_n \overset{\star}{\rightharpoonup} u$ in L^∞, then $u_n \rightharpoonup u$ in L^p.

ii) For every $1 \le p \le +\infty$, if $u_n \to u$ in L^p, then $\|u_n\|_{L^p} \to \|u\|_{L^p}$.

iii) For every $1 \le p < +\infty$, if $u_n \rightharpoonup u$ in L^p, then there exists a constant $M > 0$ such that $\forall n \in \mathbb{N}$ it holds $\|u_n\|_{L^p} \le M$ and in particular $\|u\|_{L^p} \le \liminf_{n \to +\infty} \|u_n\|_{L^p}$. For $p = +\infty$ the same result is true when $u_n \overset{*}{\rightharpoonup} u$ in L^∞.

iv) For every $1 \le p < +\infty$ and with $\frac{1}{p} + \frac{1}{q} = 1$, we have the following:

$$\left.\begin{aligned} u_n \rightharpoonup u \text{ in } L^p \\ v_n \to v \text{ in } L^q \end{aligned}\right\} \implies u_n v_n \rightharpoonup uv \text{ in } L^1$$

v) For every $1 < p < +\infty$, if there is a constant $M > 0$ such that $\|u_n\|_{L^p} \le M$, then there exists a subsequence (u_{n_k}) of (u_n) and a function $u \in L^p$ so that $u_{n_k} \rightharpoonup u$ in L^p. For $p = +\infty$ the same result is true and we have $u_{n_k} \overset{*}{\rightharpoonup} u$ in L^∞.

vi) For every $1 \le p \le +\infty$, if $u_n \to u$ in L^p, then there exists a subsequence (u_{n_k}) of (u_n) and a function $g \in L^p$ so that $u_{n_k} \to u$ pointwise almost everywhere in Ω and $|u_{n_k}| \le g$ almost everywhere in Ω.

Remark. *i) The first three properties depict the difference between strong and weak convergence: The former gives the sense of continuity to our norm, while the latter the sense of lower semi-continuity.*

ii) Property (iv) *doesn't, in fact, need Ω be bounded. Moreover, this result is false when we only have $v_n \rightharpoonup v$ in L^q.*

iii) Notice that property (v) *is the equivalent of the Bolzano-Weierstrass Theorem which in the L^p spaces is true only in the weak sense. L^1, however, fails to have this property and this is due to the fact that it is not a reflexive space.*

Observe that the above proposition tells us that we don't generally have functions which are weakly continuous. In particular, if two sequences converge, their product doesn't necessarily converges to the product of their limits - always speaking in the weak sense. However, we do have some rare examples of such functions, but we will see them later in (cf. Lemma 1.30) after we have established the a suitable framework.

It comes naturally for one to wonder how we can get weakly convergent sequences that do not converge strongly. The following theorem of Riemann and Lebesgue provides us with such a construction.

Theorem 1.10 (Riemann-Lebesgue Theorem). *Let* $\Omega = \prod_{i=1}^{n}(a_i, b_i) = (a_1, b_1) \times \cdots \times (a_n, b_n)$ *and* $u \in L^p(\Omega)$ *where* $1 \leq p \leq +\infty$. *We extend* u *periodically from* Ω *to the whole* R^n *and set*

$$u_k(x) = u(kx) \quad \text{and} \quad \overline{u} = \frac{1}{\mathrm{m}(\Omega)} \int_\Omega u(x)\, dx$$

where $\mathrm{m}(\Omega)$ *is the Lebesgue measure of* Ω. *It holds, then, that* $u_k \rightharpoonup \overline{u}$ *in* L^p, *for every* $1 \leq p < +\infty$ *and* $u_k \overset{*}{\rightharpoonup} \overline{u}$ *in* L^∞, *for* $p = +\infty$.

Now we present a lemma that will play a major role during our study of the classical methods in Chapter 2.

Lemma 1.11. *Let* $u, v \in C([a, b])$ *such that*

$$\int_a^b \big(u(x)\varphi(x) + v(x)\varphi'(x)\big)\, dx = 0, \qquad \forall \varphi \in C_0^\infty(a, b)$$

Then, v *is differentiable in* (a, b) *with derivative* $v' = u$.

Proof. Since u is continuous in $[a, b]$, we have that for every $\varphi \in C_0^\infty(a, b)$ it holds

$$0 = \int_a^b \big(u(x)\varphi(x)+v(x)\varphi'(x)\big)\, dx = \int_a^b \big((\textstyle\int_a^x u(t)\, dt)'\varphi(x)+v(x)\varphi'(x)\big)\, dx = \int_a^b (v(x) - \textstyle\int_a^x u(t)\, dt)\varphi'(x)\, dx$$

So, if we denote

$$h(x) = v(x) - \int_a^x u(t)\, dt$$

it suffices to prove that whenever h is continuous then the relation

$$\int_a^b h(x)\varphi'(x)\, dx = 0, \qquad \forall \varphi \in C_0^\infty(a, b)$$

implies that h is constant in (a, b). Indeed, observe that the function h is continuous, hence from Weierstrass' Theorem there exists a sequence of polynomials (p_n) such that $p_n \to h$ uniformly and if we set

$$c_n = \frac{1}{b-a} \int_a^b p_n(t)\, dt \quad \text{and} \quad \varphi_n(x) = \int_a^x \big(p_n(t) - c_n\big)\, dt$$

then we take $c_n \to \frac{1}{b-a}\int_a^b h(t)\,dt \equiv c$, $(p_n - c_n) \to (h - c)$ uniformly and for every $n \in \mathbb{N}$ the functions φ_n live in $C_0^\infty(a,b)$. This implies that

$$\int_a^b h(x)\varphi_n'(x)\,dx = 0, \qquad \forall n \in \mathbb{N}$$

and, since the above convergence is uniform, we have

$$0 = \int_a^b h(x)\varphi_n'(x)\,dx = \int_a^b (h(x) - c_n)\varphi_n'(x)\,dx = \int_a^b (h(x) - c_n)(p_n(x) - c_n)\,dx \to \int_a^b (h(x) - c)^2\,dx$$

$$\implies \int_a^b (h(x) - c)^2\,dx = 0 \implies h(x) = c \implies v(x) = \int_a^x u(t)\,dt + c, \quad \forall x \in [a,b]$$

and the desired result follows. □

Finally, we give a definition that will help us state the Fundamental Lemma of the Calculus of Variations and after that we present a number of corollaries and other useful results. The reader should immediately observe that some proofs are missing... These are left as an exercise.

Definition 1.12. *Let $\Omega \subset \mathbb{R}^n$ be open and $1 \leq p \leq +\infty$. We will write $u \in L^p_{loc}(\Omega)$ and say that lives* locally *in L^p if $u \in L^p(F)$ for every open set compactly contained in Ω, that is for every open F with $\overline{F} \subset \Omega$.*

Theorem 1.13 (Fundamental Lemma of the Calculus of Variations)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and $u \in L^1_{loc}(\Omega)$ such that*

$$\int_\Omega u(x)\varphi(x)\,dx = 0, \qquad \forall \varphi \in C_0^\infty(\Omega)$$

Then, $u = 0$ almost everywhere in Ω.

Corollary 1.14. *Let $\Omega \subset \mathbb{R}^n$ be open and $u \in L^1_{loc}(\Omega)$ such that*

$$\int_\Omega u(x)\varphi(x)\,dx = 0, \qquad \forall \varphi \in C_0^\infty(\Omega) \quad \textit{with} \quad \int_\Omega \varphi(x)\,dx = 0$$

Then, $u = \text{constant}$ almost everywhere in Ω.

Corollary 1.15. *Let $u \in L^1_{loc}(a,b)$ such that*

$$\int_a^b u(x)\varphi'(x)\,dx = 0, \qquad \forall \varphi \in C_0^\infty(a,b)$$

Then, $u = \text{constant}$ almost everywhere in (a,b).

Corollary 1.16. *Let $u, v \in L^1_{loc}(a,b)$ such that*

$$\int_a^b \big(u(x)\varphi(x) + v(x)\varphi'(x)\big)\,dx = 0, \qquad \forall \varphi \in C_0^\infty(a,b)$$

Then, v is differentiable almost everywhere in (a,b) with derivative $v' = u$ almost everywhere.

Remark. *Notice that Corollary 1.16 is a generalization of Lemma 1.11. The generalization in higher dimensions, however, is not an easy task and problems in $\mathbb{R}^n$, with $n > 1$, require a different approach from that in one dimension.*

Theorem 1.17. *Let $\Omega \subset \mathbb{R}^n$ be open and $u \in L^1(\Omega)$. Then, for every $\varepsilon > 0$ there exists $\delta > 0$ so that for any measurable set $E \subset \Omega$ it holds that*

$$\mathrm{m}(E) \leq \delta \implies \int_E |u(x)|\, dx \leq \varepsilon$$

1.4 Sobolev Spaces

Sobolev spaces were created in order to provide a notion of integration by parts even for functions that are not differentiable. Before we see these spaces themselves we have to make clear what we mean "integrating by parts" for such functions. So, we begin with the definition of the *weak derivative*.

Definition 1.18. *Let $\Omega \subset \mathbb{R}^n$ be open and $u \in L^1_{loc}(\Omega)$. We say that the function $u \in L^1_{loc}(\Omega)$ is the* weak partial derivative *of u with respect to x_i if v satisfies*

$$\int_\Omega v(x)\varphi(x)\, dx = -\int_\Omega u(x)\frac{\partial \varphi}{\partial x_i}(x)\, dx, \qquad \forall \varphi \in C_0^\infty(\Omega)$$

If the weak partial derivatives of u exists for all $i = 1, 2, \ldots, n$, then we say that that u is weakly differentiable.

The Fundamental Lemma of Calculus of Variations 1.13 tells us that if the weak partial derivative of u with respect to x_i exists, it is unique (almost everywhere) and so, abusing the standard notation, we will denote it with $u_{x_i} = \frac{\partial u}{\partial x_i}$. Similarly, we say that $(u_{x_1}, u_{x_2}, \ldots, u_{x_n})$ is the weak derivative *of u and we will denote it with ∇u.*

Remark. *i) The definition is similar if we want to introduce weak derivatives of higher order.*

ii) For a C^1 function the weak derivative coincides with the usual pointwise derivative.

iii) All the usual rules of differentiation can be generalized without any problem in our case of weak derivatives.

However generalized this notion of weak derivative may be, we still fail to have derivatives for all the "relatively smooth" functions we can think of. For instance, the discontinuous functions in $\mathbb{R}$ don't have weak derivatives.

Example (Dirac Mass)**.** *The characteristic function of $(0, +\infty)$*

$$\mathcal{X}_{(0,+\infty)}(x) = \begin{cases} 1, & for\ x > 0 \\ 0, & for\ x \leq 0 \end{cases}$$

has no weak derivative. Indeed, assume that δ is its weak derivative. Then, let $\Omega = (-1, 1)$ and $\psi \in C_0^\infty(0,1)$ be arbitrary. If ψ is extended to be $\psi(x) = 0$ in $(-1, 0)$, then $\psi \in C_0^\infty(-1,1)$ and by definition we have

$$\int_{-1}^1 \delta(x)\psi(x)\, dx = -\int_{-1}^1 \mathcal{X}_{(0,1]}(x)\psi'(x)\, dx = -\int_0^1 \psi'(x)\, dx = \psi(0) - \psi(1) = 0$$

$$\implies \int_0^1 \delta(x)\psi(x)\,dx = \int_{-1}^1 \delta(x)\psi(x)\,dx = 0, \qquad \forall \psi \in C_0^\infty(0,1)$$

However, the Fundamental Lemma of Calculus of Variations 1.13 tells us that $\delta = 0$ almost everywhere in $(0,1)$ and similarly we can deduce that $\delta = 0$ almost everywhere in $(-1,0)$. So, if $\varphi \in C_0^\infty(-1,1)$, we now have

$$0 = \int_{-1}^1 \delta(x)\varphi(x)\,dx = -\int_{-1}^1 \mathcal{X}_{(0,1]}(x)\varphi'(x)\,dx = -\int_0^1 \varphi'(x)\,dx = \varphi(0) - \varphi(1) = \varphi(0)$$

Thus, $\varphi(0) = 0$ for every function in $C_0^\infty(-1,1)$ which is clearly absurd.

Now we are ready to properly introduce the spaces with which we are going to work.

Definition 1.19. *Let $\Omega \subset \mathbb{R}^n$ be open and $1 \le p \le +\infty$.*

i) *We will denote with $W^{1,p}(\Omega)$ the set all weakly differentiable functions $u : \Omega \to \mathbb{R}$ who live in $L^p(\Omega)$ and whose weak partial derivatives live also in $L^p(\Omega)$. That is, $u \in W^{1,p}(\Omega)$ whenever u is weakly differentiable, $u \in L^p(\Omega)$ and $u_{x_i} \in L^p(\Omega)$ for all the weak partial derivatives u_{x_i}, $i = 1, 2, \ldots, n$, of u. This space endowed with the norm*

$$\|u\|_{W^{1,p}} = (\|u\|_{L^p}^p + \|\nabla u\|_{L^p}^p)^{\frac{1}{p}}, \quad \text{for } 1 \le p < +\infty$$

$$\|u\|_{W^{1,\infty}} = \max\{\|u\|_{L^\infty}, \|\nabla u\|_{L^\infty}\}, \quad \text{for } p = +\infty$$

is a Banach space and will be called Sobolev space of first order, *or simply* Sobolev space.

ii) *Similarly, $W^{1,p}(\Omega; \mathbb{R}^m)$ will be the set of all maps $u = (u^1, \ldots, u^m) : \Omega \to \mathbb{R}^m$ with $u^j \in W^{1,p}(\Omega)$ for every $j = 1, \ldots, m$.*

iii) *With $W_0^{1,p}(\Omega)$ we will denote the closure of all $C_0^\infty(\Omega)$ functions within $W^{1,p}(\Omega)$. When Ω is bounded we can say that $W_0^{1,p}(\Omega)$ is the set of all functions $u \in W^{1,p}(\Omega)$ with $u = 0$ on $\partial\Omega$ or equivalently it is the set of all functions in $W^{1,p}(\Omega)$ with compact support.*

iv) *If $u_0 \in W^{1,p}(\Omega)$, we will also use the notation $u_0 + W_0^{1,p}(\Omega)$ for all the functions $u \in W^{1,p}(\Omega)$ with $u - u_0 \in W_0^{1,p}(\Omega)$. When Ω is bounded this means that $u = u_0$ on $\partial\Omega$.*

v) $W_0^{1,\infty}(\Omega) = W^{1,\infty}(\Omega) \cap W_0^{1,1}(\Omega)$

vi) *We can define in the same way the Sobolev spaces of higher order. In particular, if $k \ge 0$ is an integer, we denote with $W^{k,p}(\Omega)$ the set of all functions $u : \Omega \to \mathbb{R}$ that are k times weakly differentiable and whose weak partial derivatives $D^\alpha u \in L^p(\Omega)$ for every multi-index $a \in \mathcal{A}_m$ and for every integer $0 \le m \le k$. This* Sobolev space of k order *is a Banach space when we attach the norm*

$$\|u\|_{W^{k,p}} = \left(\sum_{0 \le |a| \le k} \|D^a u\|_{L^p}^p \right)^{\frac{1}{p}}, \quad \text{for } 1 \le p < +\infty$$

$$\|u\|_{W^{k,\infty}} = \max_{0 \le |a| \le k} \{\|D^a u\|_{L^\infty}\}, \quad \text{for } p = +\infty$$

By abuse of notation we will write $W^{0,p} = L^p$.

vii) For $1 \le p < +\infty$, *the space* $W_0^{k,p}(\Omega)$ *will denote the closure of* $C_0^\infty(\Omega)$ *in* $W^{k,p}(\Omega)$. *For* $p = +\infty$, $W_0^{k,\infty}(\Omega) = W^{k,\infty}(\Omega) \cap W_0^{k,1}(\Omega)$.

Remark. *i) In the case* $p = 2$ *the spaces* $W^{1,2}$, $W_0^{1,2}$, $W^{k,2}$ *and* $W_0^{k,2}$ *are sometimes denoted with* H^1, H_0^1, H^k *and* H_0^k *respectively and are known as* Hilbert spaces.

ii) Note that for every $1 \le p < +\infty$ *and for a bounded set* $\Omega \subset \mathbb{R}^n$ *we have*

$$C^1(\overline{\Omega}) \subsetneq W^{1,\infty}(\Omega) \subsetneq W^{1,p}(\Omega) \subsetneq L^p(\Omega)$$

Theorem 1.20. *Let* $\Omega \subset \mathbb{R}^n$ *be open,* $k \ge 1$ *be an integer and* $1 \le p \le +\infty$.

i) The Banach space $W^{k,p}(\Omega)$ *is separable for every* $1 \le p < +\infty$ *and reflexive, for* $1 < p < +\infty$.

ii) $W^{1,2}$ *is a Hilbert space with inner product*

$$\langle u\,; v\rangle_{W^{1,2}} = \int_\Omega u(x)v(x)\,dx + \int_\Omega \langle \nabla u(x)\,; \nabla v(x)\rangle\,dx$$

iii) For every $1 \le p < +\infty$, *the functions in* $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ *are dense in* $W^{k,p}(\Omega)$. *Furthermore, if* Ω *is a bounded domain with Lipschitz boundary (cf. Definition 1.23), then* $C^\infty(\overline{\Omega})$ *is also dense in* $W^{k,p}(\Omega)$.

iv) For $1 \le p < +\infty$, *we have that* $W_0^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n)$.

The following theorem provides us with criteria with which we can decide whether a function belongs in $W^{1,p}$ or not.

Theorem 1.21. *Let* $\Omega \subset \mathbb{R}^n$ *be open,* $1 < p \le +\infty$ *and* $q \in \mathbb{R}$ *be such that* $\frac{1}{p} + \frac{1}{q} = 1$. *Suppose we have a function* $u \in L^p(\Omega)$. *Then, the following are equivalent.*

i) $u \in W^{1,p}(\Omega)$

ii) There exists a constant $c = c(u, \Omega, p)$ *such that*

$$\left| \int_\Omega u(x) \frac{\partial \varphi}{\partial x_i}(x)\,dx \right| \le c\,\|\varphi\|_{L^q(\Omega)}\,, \qquad \forall \varphi \in C_0^\infty(\Omega), \quad i = 1, 2, \ldots, n$$

iii) There exists a constant $c = c(u, \Omega, p)$ *such that for every open set* F *compactly contained in* Ω *and for every* $h \in \mathbb{R}^n$ *with* $|h| < \operatorname{dist}(F, \Omega^c)$ *it holds*

$$\left(\int_F |u(x+h) - u(x)|^p\,dx \right)^{\frac{1}{p}} \le c|h|, \quad \textit{for } 1 < p < +\infty$$

$$|u(x+h) - u(x)| \le c|h|, \quad \textit{for } p = +\infty$$

The constant c *can be chosen to be* $c = \|\nabla u\|_{L^p}$.

Remark. *i) In the case when* $p = 1$ *we only have* (i) $\implies$ (ii) $\iff$ (iii). *The functions satisfying* (ii) *and* (iii) *are known as functions of bounded variations.*

ii) From the theorem we can see that generally $C^{0,1}(\overline{\Omega}) \subsetneq W^{1,\infty}$*. However, when* Ω *is additionally convex, then we can identify* $W^{0,1}(\Omega)$ *with the set of all Lipschitz continuous functions in* Ω*.*

The truth, however, is that we will not actually work on Sobolev spaces as they are. From now on, when we write an equality between two functions $u, v \in W^{k,p}(\Omega)$, i.e. $u = v$, we mean that "$u = v$ *almost everywhere in* Ω", even though at cases we may restate this clarification regardless. To be precise, writing $u \in W^{k,p}(\Omega)$ implies that u belongs to an equivalence class of the quotient space of $W^{k,p}(\Omega)$ with equivalence relation "=", which means *equality almost everywhere*. Then, u is in fact a representative of its class and explicitly stating that "$u = v$ almost everywhere" suggests that we have just chosen a specific representative. In the rest of the text we will not really pay attention to these details, nevertheless they are important for better understanding our theory.

The following lemma gives us a way to chose such a representative when working in one dimension.

Lemma 1.22. *Let* $u \in W^{1,p}(a,b)$*, where* $1 \leq p \leq +\infty$*. Then, there exists a continuous function* $\tilde{u} \in C([a,b])$ *such that* $\tilde{u} = u$ *almost everywhere in* (a,b) *and it holds that*

$$\tilde{u}(y) - \tilde{u}(x) = \int_x^y u'(x)\, dx, \qquad \forall x.y \in [a,b]$$

Remark. *i) This lemma is a particular case of the Sobolev Embedding Theorem that we will present shortly. Hence, we have that*

$$C^1([a,b]) \subset W^{1,p}(a,b) \subset C([a,b]), \quad for\ 1 \leq p \leq +\infty$$

ii) In fact, if $u \in W^{1,p}(a,b)$*, for* $1 < p < +\infty$*, then* $u \in C^{1,\frac{1}{q}}([a,b])$*, where* $\frac{1}{p} + \frac{1}{q} = 1$ *meaning that* u *is* $\frac{1}{q}$*-Hölder continuous. Moreover, in the previous remark we already saw that we can identify* $W^{1,\infty}(a,b)$ *with* $C^{0,1}([a,b])$*.*

We can now proceed to some really important theorems of this field regarding inclusions and embeddings. So, to begin with, we put ourselves into the correct framework.

Definition 1.23. *Let* $\Omega \subset \mathbb{R}^n$ *be bounded and open and* $k \geq 1$ *be an integer.*

i) We say that Ω *is a* bounded open set with C^k boundary *if for every point* $x \in \partial\Omega$*, there exists a neighbourhood* $U \subset \mathbb{R}^n$ *of* x *and an 1-1 and onto map* $H : Q \to U$*, where* Q *is the open* n*-dimensional cube, that is* $Q = \{x \in \mathbb{R}^n \ : \ |x_i| < 1, \quad for\ i = 1, 2, \ldots, n\}$*. The map* H *must also satisfy the following*

$$H \in C^k(\overline{\Omega}), \qquad H^{-1} \in C^k(\overline{U})$$

$$H(Q_+) = U \cap \Omega, \qquad H(Q_0) = U \cap \partial\Omega$$

where $Q_+ = \{x \in Q \ : \ x_n > 0\}$ *and* $Q_0 = \{x \in Q \ : \ x_n = 0\}$*.*

ii) If H *is a* $C^{k,\alpha}$ *function with* $0 < \alpha \leq 1$*, we say that* Ω *is a* bounded open set with $C^{k,\alpha}$ boundary.

iii) If H *is only in* $C^{0,1}$*, we say that* Ω *is a* bounded open set with Lipschitz boundary.

Theorem 1.24 (Sobolev Embedding Theorem)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set with Lipschitz boundary and* $1 \leq p \leq +\infty$*. We distinguish between three different cases:*

$1 \le p < n$) *If* $\overline{p}$ *is such that* $\frac{1}{p} + \frac{1}{\overline{p}} = \frac{1}{n}$, *or equivalently* $\overline{p} = \frac{np}{n-p}$, *then for every* $1 \le q \le \overline{p}$ *it holds*

$$W^{1,p}(\Omega) \subset L^q(\Omega)$$

In particular, for every $q \in [1, \overline{p}]$ *there exists a constant* $c = c(\Omega, p, q) > 0$ *so that*

$$\|u\|_{L^q} \le c\,\|u\|_{W^{1,p}}$$

$p = n$) *For every* $1 \le q < +\infty$ *we have*

$$W^{1,n}(\Omega) \subset L^q(\Omega)$$

In particular, for every $q \in [1, +\infty)$ *there exists a constant* $c = c(\Omega, p, q) > 0$ *so that*

$$\|u\|_{L^q} \le c\,\|u\|_{W^{1,n}}$$

$p > n$) *For every* $0 \le \alpha \le 1 - \frac{n}{p}$ *we get*

$$W^{1,p}(\Omega) \subset C^{0,\alpha}(\Omega)$$

In particular, there exists a constant $c = c(\Omega, p) > 0$ *so that*

$$\|u\|_{L^\infty} \le c\,\|u\|_{W^{1,p}}$$

Theorem 1.25 (Rellich-Kondrachov Theorem)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set with Lipschitz boundary and* $1 \le p \le +\infty$. *We distinguish between three different cases:*

$1 \le p < n$) *For every* $q \in [1, \overline{p})$, *where* $\overline{p}$ *as in the preceding theorem, the embedding of* $W^{1,p}$ *in* L^q *is compact. That is, any bounded set of* $W^{1,p}$ *is precompact in* L^q, *i.e. its closure is compact and it is in* L^q. *This result is false in the case when* $q = \overline{p}$.

$p = n$) *For every* $q \in [1, +\infty)$ *the embedding of* $W^{1,p}$ *in* L^q *is compact.*

$p > n$) *For every* $\alpha \in [1, 1 - \frac{n}{p}]$ *the embedding of* $W^{1,p}(\Omega)$ *in* $C^{0,\alpha}(\overline{\Omega})$ *is compact.*

Observe that for every $1 \le p \le +\infty$ *the embedding of* $W^{1,p}(\Omega)$ *in* $L^p(\Omega)$ *is compact.*

Remark. *i) We can obtain similar embeddings for spaces of higher order* $W^{k,p}$.

ii) If the set Ω *is not bounded, for example if* $\Omega = \mathbb{R}^n$, *the compactness of our embeddings is lost.*

iii) If we consider the spaces $W_0^{1,p}$ *instead of* $W^{1,p}$, *we find that the same embeddings apply but without any restrictions on the boundary* $\partial\Omega$ *anymore.*

iv) the case $n = 1$) *Here we don't have the case when* $1 \le p < n$, *but we do have the results from Lemma 1.22. Summarizing for when* $\Omega = (a, b)$ *and* $\frac{1}{p} + \frac{1}{q} = 1$ *we have*

$$\begin{aligned} C_0^\infty(a,b) \subset \cdots \subset W^{k,p}(a,b) \subset \cdots \subset W^{2,p}(a,b) \subset C^1([a,b]) \subset W^{1,p}(a,b) \\ \subset C^{0,\frac{1}{q}}([a,b]) \subset C([a,b]) \subset L^\infty(a,b) \subset \cdots \subset L^p(a,b) \subset \cdots \subset L^1(a,b) \end{aligned}$$

and the C_0^∞ *functions are dense in* L^1 *with the usual norm for each space. The compactness of our embeddings in the dimension* $n = 1$ *results from the Ascoli-Arzela Theorem 1.2.*

Corollary 1.26. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and $1 < p < +\infty$. If $u \in W^{1,p}(\Omega)$ and $\varphi \in W_0^{1,q}(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$, then for every $i = 1, 2, \ldots, n$ it holds that*

$$\int_\Omega u_{x_i}(x)\varphi(x)\,dx = -\int_\Omega u(x)\varphi_{x_i}(x)\,dx$$

One of the major consequences of the Rellich-Kondrachov Theorem is that it translates the weak convergence of a sequence in $W^{1,p}$ into strong convergence in L^p.

Definition 1.27. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary.*

i) If $1 \leq p < +\infty$, we will say that a sequence (u_n) converges weakly to *u* in *$W^{1,p}(\Omega)$ when $u_n, u \in W^{1,p}$ and it holds that*

$$u_n \rightharpoonup u \text{ in } L^p \quad \text{and} \quad \nabla u_n \rightharpoonup \nabla u \text{ in } L^p$$

We will write, then, "$u_n \rightharpoonup u$ in $W^{1,p}$" (or sometimes $u_n \overset{W^{1,p}}{\rightharpoonup} u$).

ii) If $p = +\infty$ we will say that a sequence (u_n) weakly converges to *u* in *$W^{1,\infty}(\Omega)$ when $u_n, u \in W^{1,\infty}$ and it holds*

$$u_n \rightharpoonup u \text{ in } L^\infty \quad \text{and} \quad \nabla u_n \rightharpoonup \nabla u \text{ in } L^\infty$$

We will write, then, "$u_n \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}$"

Corollary 1.28. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary.*

i) If $1 \leq p < +\infty$ and $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$, then $u_n \to u$ in $L^p(\Omega)$.

ii) If $p = +\infty$ and $u_n \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}(\Omega)$, then $u_n \to u$ in $L^\infty(\Omega)$.

The next theorem is a consequence of Proposition 1.9 we encountered earlier.

Theorem 1.29. *Suppose $1 < p < +\infty$ and that $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary. If for the sequence (u_n) in $W^{1,p}(\Omega)$ there exists a constant $c > 0$ such that*

$$\|u_n\|_{W^{1,p}} \leq c \quad \text{for every } n \in \mathbb{N}$$

then there exists a subsequence (u_{n_k}) of (u_n) and a function $u \in W^{1,p}(\Omega)$ so that

$$u_{n_k} \rightharpoonup u \text{ in } W^{1,p}(\Omega)$$

And here there is another interesting result coming from studying the same proposition:

Lemma 1.30. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with Lipschitz boundary and $p > 2$. Suppose that the sequence $(u_n) = \big((\chi_n, \psi_n)\big)$ is in $W^{1,p}(\Omega; \mathbb{R}^2)$ and that*

$$u_n = (\chi_n, \psi_n) \rightharpoonup u = (\chi, y) \text{ in } W^{1,p}(\Omega; \mathbb{R}^2)$$

Then, it holds that

$$\det \nabla u_n \rightharpoonup \det \nabla u \text{ in } L^{p/2}(\Omega)$$

Finally, we present the last tool regarding Sobolev spaces that we will need throughout our study. This will prove to be particularly useful.

Theorem 1.31 (Poincaré Inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $1 \le p \le +\infty$. Then, there exists constant $c = c(\Omega, p) > 0$ so that*

$$\|u\|_{L^p} \le c \|\nabla u\|_{L^p}, \qquad \forall u \in W_0^{1,p}(\Omega)$$

Equivalently, there is a constant $\tilde{c} = \tilde{c}(\Omega, p)$ so that

$$\|u\|_{W^{1,p}} \le \tilde{c} \|u\|_{L^p}, \qquad \forall u \in W_0^{1,p}(\Omega)$$

Remark. *i) The necessity of being into the space $W_0^{1,p}$ instead of $W^{1,p}$ comes from the constant functions living in $W^{1,p}$ which we want to avoid.*

ii) If $\Omega \subset \mathbb{R}^n$ happens to be a bounded connected open set with Lipschitz boundary and we denote

$$u_\Omega = \frac{1}{\mathrm{m}(\Omega)} \int_\Omega u(x)\, dx$$

then one may encounter Poincaré Inequality into the following form: There exists a constant $c = c(\Omega, p) > 0$ so that

$$\|u - u_\Omega\|_{L^p} \le c \|\nabla u\|_{L^p}, \qquad \forall u \in W^{1,p}(\Omega)$$

iii) Yet another interesting form of this inequality comes from working with functions living in $u_0 + W_0^{1,p}(\Omega)$, where $u_0 \in W^{1,p}(\Omega)$. Then, there exist constants $c_1, c_2 > 0$ so that

$$c_1 \|u\|_{W^{1,p}} - c_2 \|u_0\|_{W^{1,p}} \le \|\nabla u\|_{L^p}, \qquad \forall u \in u_0 + W_0^{1,p}(\Omega)$$

It is this last form of the equality the one that we are going to need later.

1.5 Convex Analysis

In this final preliminary section we are going to present some results regarding convex functions that play a surprisingly important role in the theorems of this field.

Definition 1.32. *i) The set $\Omega \subset \mathbb{R}^n$ is called* convex *when for every $x, y \in \Omega$ and for every $\lambda \in [0, 1]$ we have that $\lambda x + (1 - \lambda) y \in \Omega$.*

ii) Let $\Omega \subset \mathbb{R}^n$ be convex. A function $f : \Omega \to \mathbb{R}$ is called convex *when for every $x, y \in \Omega$ and for every $\lambda \in [0, 1]$ we have that*

$$f(\lambda x + (1 - \lambda) y) \le \lambda f(x) + (1 - \lambda) f(y)$$

From now on whenever say that a function is convex we will silently assume that its domain is also convex even if we don't state it explicitly.

Theorem 1.33. *Let $f : \mathbb{R}^n \to \mathbb{R}$ live in $C^1(\mathbb{R}^n)$.*

i) The function f is convex if, and only if,

$$f(x) \geq f(y) + \langle \nabla f(y) \, ; x - y \rangle \,, \qquad \forall x, y \in \mathbb{R}^n$$

ii) If f is additionally $C^2(\mathbb{R}^n)$, then it is convex if, and only if, its Hessian, Δf, is positive semi-definite.

The above theorem provides us with some criteria equivalent to convexity. And bellow we have a classic result regarding convex functions.

Theorem 1.34. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $\overline{\Omega}$ be convex. If $f : \overline{\Omega} \to \mathbb{R}$ is convex, then f is continuous in Ω, i.e. the interior of $\overline{\Omega}$.*

Also, we cannot forget the famous Jensen inequality which will come in handy later in our study.

Theorem 1.35 (Jensen Inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and the function $f : \mathbb{R} \to \mathbb{R}$ be convex. Then, for every $u \in L^1(\Omega)$ it holds that*

$$f\left(\frac{1}{\mathrm{m}(\Omega)} \int_\Omega u(x)\, dx\right) \leq \frac{1}{\mathrm{m}(\Omega)} \int_\Omega f(u(x))\, dx$$

Now we need to present a new notion that will help us "force" the existence of solutions to some evasive problems we will encounter. Observe that in our new definition we allow not only finite functions.

Definition 1.36. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$.*

i) The dual *of f, also known as the* Legendre transform *of f, is the function $f^\star : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ given by the formula*

$$f^\star(y) = \sup_{x \in \mathbb{R}^n} \{\langle x \, ; y \rangle - f(x)\}$$

ii) The bidual *of f, also known as the* second Legendre transform *of f, is the function $f^{\star\star} : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ given by the formula*

$$f^{\star\star}(x) = \sup_{y \in \mathbb{R}^n} \{\langle y \, ; x \rangle - f^\star(y)\}$$

We have now a theorem that gives us some properties of the Legendre transform and also helps us identify the points that "ruin" the convexity of our functions.

Theorem 1.37. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$.*

i) The function $f^\star$ is convex, even if f itself is not.

ii) The function $f^{\star\star}$ is convex (from (i)*) and it holds that $f^{\star\star} \leq f$. If f is additionally convex and finite, then $f^{\star\star} = f$. The function $f^{\star\star}$ is the so-called* convex envelop *of f, that is the largest convex function smaller than f.*

iii) $f^{\star\star\star} = f^\star$

iv) If f is convex, finite and lives in $C^1(\mathbb{R}^n)$, then

$$f(x) + f^\star(\nabla f(x)) = \langle \nabla f(x)\,;x\rangle\,, \qquad \forall x \in \mathbb{R}^n$$

v) If $f : \mathbb{R}^n \to \mathbb{R}$ is convex and it holds that

$$\lim_{|x|\to+\infty} \frac{f(x)}{|x|} = +\infty$$

then $f^\star$ is everywhere finite.

vi) If $f : \mathbb{R}^n \to \mathbb{R}$ is strictly convex and it holds that

$$\lim_{|x|\to+\infty} \frac{f(x)}{|x|} = +\infty$$

then $f^\star \in C^1(\mathbb{R}^n)$. If f lives additionally in $C^1(\mathbb{R}^n)$ and

$$f(x) + f^\star(y) = \langle y\,;x\rangle$$

then we have that

$$y = \nabla f(x) \quad \textit{and} \quad x = \nabla f^\star(y)$$

This last statement implies that ∇f is invertible with $\nabla f^\star$ as its inverse.

The following an example tells that the condition of (v) is necessary for the dual of a function to be finite.

Example. *Suppose we have the function $f : \mathbb{R} \to \mathbb{R}$ with $f(\xi) = \sqrt{1+\xi^2}$. Then, $f'' > 0$ and so f is convex, while $\frac{1}{|\xi|} f(\xi) \to 1$. We also find that the dual of f is*

$$f^\star(v) = \sup_{\xi\in\mathbb{R}} \{v\xi - f(\xi)\} = \begin{cases} -\sqrt{1-v^2}, & |v| \le 1 \\ +\infty, & |v| > 1 \end{cases}$$

and so $f^\star$ is not everywhere finite.

Finally, we present an alternative way to produce the bidual of a function.

Theorem 1.38 (Carathéodory Theorem)**.** *If $f : \mathbb{R}^n \to \mathbb{R}$, then*

$$f^{\star\star}(x) = \inf\left\{ \sum_{i=1}^{n+1} \lambda_i f(x_i) \ : \ x = \sum_{i=1}^{n+1} \lambda_i x_i \quad \textit{where} \quad \lambda_i \ge 0 \quad \textit{and} \quad \sum_{i=1}^{n+1} \lambda_i = 1 \right\}$$

Chapter 2

Classical methods

2.1 Introduction

Our main purpose is to find the minimizers, if any, of

$$\inf \left\{ I(u) = \int_a^b f\left(x, u(x), u'(x)\right) \, dx \ : \ u \in X \right\} = m$$

and determine, whenever possible, how regular they are. Our function f is integrable on (a, b) and X is space of all admissible functions. In this chapter we will study a simpler case of the problem when

$$X = \{u \in C^1([a, b]) \ : \ u(a) = \alpha \ \ and \ \ u(b) = \beta\}$$

and $f \in C^1([a, b] \times \mathbb{R} \times \mathbb{R})$. The title of this chapter depicts exactly the *classical* approach we follow for the time being which is based on the standard practice of finding the derivative of our function and pinpointing its roots that may in turn be minimizers or maximizers, global or local, or even saddle points.

For starters, we present the equivalent of this standard method with the *Euler-Lagrange equation*, namely

$$\frac{d}{dx} f_\xi \left(x, u(x), u'(x)\right) = f_u \left(x, u(x), u'(x)\right), \qquad x \in [a, b] \tag{E}$$

playing the role of $I'(u)$. Its roots are known as the *stationary points* of I which are not necessarily minima of $I(u)$ as one naturally expects. However, if the function $(u, \xi) \to f(x, u, \xi)$ is convex we, indeed, have that these roots are minimizers of $I(u)$. Immediately after, we give an alternative form of the above equation from which we can derive the so-called *first integral* of (E). Finally, we see that (E) is equivalent with a system of partial differential equations whose solutions can be found through solving another partial differential equation that we study afterwards.

2.2 The Euler-Lagrange Equation

We begin with our main result:

Theorem 2.1. *Let* $f \in C^1([a,b] \times \mathbb{R} \times \mathbb{R})$ *and*

$$\inf_{u \in X} \left\{ I(u) = \int_a^b f\left(x, u(x), u'(x)\right)\, dx \right\} = m \tag{P}$$

where $X = \{u \in C^1([a,b]) \ : \ u(a) = \alpha \ \text{ and } \ u(b) = \beta\}$.

i) If (P) *admits a minimizer, say* $\overline{u} \in X$*, then* $\overline{u}$ *solves the Euler-Lagrange equation, i.e.*

$$\frac{d}{dx} f_\xi\left(x, \overline{u}(x), \overline{u}'(x)\right) = f_u\left(x, \overline{u}(x), \overline{u}'(x)\right), \qquad \forall x \in (a,b) \tag{2.1}$$

ii) Conversely, if f *is additionally* C^1*, the function* $(u,\xi) \to f(x,u,\xi)$ *is convex for every* $x \in [a,b]$ *and* $\overline{u}$ *satisfies* (E)*, then* $\overline{u}$ *is a minimizer of* (P)*.*

iii) Finally, when the function $(u,\xi) \to f(x,u,\xi)$ *is* strictly convex *for every* $x \in [a,b]$*, then a minimizer of* (P) *is unique if it exists.*

Remark. *i) Note that this theorem does not provide any information on the existence of a minimizer.*

ii) If the function $(u,\xi) \to f(x,u,\xi)$ *is not convex, then a solution of the E-L equation is not always a (absolute) minimizer of* (P)*. These solutions are called* stationary points *of* I *and they can be local minimizers or maximizers or saddle points.*

iii) If the functions we consider are $u : \mathbb{R} \to \mathbb{R}^m$*, then the E-L equation is a system of* m *differential equations. Furthermore, if* $u : \mathbb{R}^n \to \mathbb{R}^m$*, then we have* m *partial differentials equations of order 2.*

iv) Pay close attention to the fact that f *does not need to be* C^2 *in order for the E-L equation to hold, provided that we have a* C^1 *minimizer. This is due to the fact that we work on only one dimension which enables us to use Lemma 1.11, as we are going to see in the proof.*

Proof. i) Let $\overline{u}$ be a minimizer of $\inf\{I(u)\} = m$. Then we can write every element $u \in X$ as $u = \overline{u} + hv$ where $h \in \mathbb{R}$ and $v \in C_0^\infty(a,b)$ are arbitrary. So if $\Phi(h) \equiv I(\overline{u} + hv)$, then $\Phi \in C^1(\mathbb{R})$ (since f *is* C^1) and we have that $\Phi(0) \leq \Phi(h)$ for every $h \in \mathbb{R}$ which implies

$$\left.\frac{d}{dh} I(\overline{u} + hv)\right|_{h=0} = \Phi'(0) = 0$$

and thus

$$\int_b^a \left(f_\xi\left(x, \overline{u}, \overline{u}'\right) v'(x) + f_u\left(x, \overline{u}, \overline{u}'\right) v(x) \right) dx = 0, \qquad \forall v \in C_0^\infty(a,b) \tag{E_w}$$

This equation is called the *weak form* of the *E-L equation*. Now, Lemma 1.11 gives us exactly (2.1) and so we have completed the proof of (i).

ii) Let $\overline{u}$ be a solution of (2.1) with $\overline{u}(a) = \alpha$ and $\overline{u}(b) = \beta$. From Theorem 1.33 since $(u, \xi) \to f(x, u, \xi)$ is convex we have that for every $u \in X$ it holds

$$f(x, u, u') \geq f(x, \overline{u}, \overline{u}') + f_u(x, \overline{u}, \overline{u}')(u - \overline{u}) + f_\xi(x, \overline{u}, \overline{u}')(u' - \overline{u}')$$

and thus by integrating and because $u(a) - \overline{u}(a) = u(b) - \overline{u}(b) = 0$ we take

$$I(u) \geq I(\overline{u}) + \int_a^b \big(f_u(x, \overline{u}, \overline{u}')(u - \overline{u}) + f_\xi(x, \overline{u}, \overline{u}')(u' - \overline{u}')\big)\, dx$$

$$\implies I(u) \geq I(\overline{u}) + \int_a^b \left(f_u(x, \overline{u}, \overline{u}') - \frac{d}{dx} f_\xi(x, \overline{u}, \overline{u}')\right)(u - \overline{u})\, dx$$

which, since (2.1) holds by hypothesis, immediately yields our desired result.

iii) Suppose that we have two minimizers of (P), $u, v \in X$. We set $w \equiv \frac{1}{2}u + \frac{1}{2}v$ and note that $w \in X$. By the convexity of $(u, \xi) \to f(x, u, \xi)$ we have

$$\frac{1}{2} f(x, u, u') + \frac{1}{2} f(x, v, v') \geq f(x, \frac{1}{2}u + \frac{1}{2}v, \frac{1}{2}u' + \frac{1}{2}v') = f(x, w, w')$$

and after integrating

$$m = \frac{1}{2} I(u) + \frac{1}{2} I(v) \geq I(w) \geq m \implies I(w) = m \tag{2.2}$$

So, w is also a minimizer of (P). From above two relations we deduce, then, that

$$\int_a^b \frac{1}{2} f(x, u, u') + \frac{1}{2} f(x, v, v') - f(x, \frac{u + u}{2}, \frac{u' + v'}{2})\, dx = \frac{1}{2} I(u) + \frac{1}{2} I(v) - I(w) = \frac{m}{2} + \frac{m}{2} - m = 0$$

while the integrand quantity is non-negative due to the inequalities in (2.2). This means that we necessarily have

$$\frac{1}{2} f(x, u, u') + \frac{1}{2} f(x, v, v') = f(x, \frac{u + u}{2}, \frac{u' + v'}{2})$$

and thus, due to the strict convexity of $(u, \xi) \to f(x, u, \xi)$, $u = v$ and $u' = v'$ in (a,b) or in other words the minimizer is unique.

□

We will see further clarifications on the proof of uniqueness at a remark we state at the end of Section 3.2.

2.3 Second form of the Euler-Lagrange equation

In this section we will see a different form of the E-L equation sometimes also called *DuBois-Reymond equation* and we will define the so-called *first integral* of the E-L equation. In particular, the equation

$$\frac{d}{dx}\big(f(x, u(x), u'(x)) - u'(x) f_\xi(x, u(x), u'(x))\big) = f_x(x, u(x), u'(x)) \tag{E'}$$

is called the *second form* of the E-L equation and its godfather is the following theorem.

Theorem 2.2. *Let $f \in C^1([a,b]\times\mathbb{R}\times\mathbb{R})$ and $X = \{u \in C^1([a,b]) \ : \ u(a)=\alpha,\ u(b)=\beta\}$ and suppose that $\overline{u} \in X$ is a minimizer of* (P)*. Then, for every $x \in [a,b]$ the function $\overline{u}$ is a solution of the equation* (E')*.*

Remark. *Note that from the previous theorem $\overline{u}$ also solves the E-L equation.*

If one further assumes that the minimizer $\overline{u}$ is C^2, then we can have a really short proof which only needs some simple calculations here and there and which is left to the reader. For we will now examine a very different proof that uses the tools responsible of the name of *Calculus of Variations*. In particular, the technique we are about to use is known as *variations of the independent variables*.

Proof. Let $\epsilon \in \mathbb{R}$ with $|\epsilon| \le 1$ and $\varphi \in C_0^\infty([a,b])$ and consider the function

$$\xi(x,\epsilon) = x + \frac{\varphi(x)}{2\left\|\varphi'\right\|_{L^\infty}}\epsilon$$

Note that ξ is differentiable while $\xi(\cdot,\epsilon)$ is 1-1 and onto $[a,b]$ since $\xi(a,\epsilon)=a,\ \xi(b,\epsilon)=b$ (recall that $\varphi \in C_0^\infty$) and $\xi_x(\cdot,\epsilon) > 0$ since

$$\xi_x(x,\epsilon) = 1 + \frac{\varphi'(x)\epsilon}{2\left\|\varphi'\right\|_{L^\infty}} \ge 1 - \frac{|\varphi'(x)\epsilon|}{2\left\|\varphi'\right\|_{L^\infty}} \ge 1 - \frac{\left\|\varphi'\right\|_{L^\infty}\cdot 1}{2\left\|\varphi'\right\|_{L^\infty}} = \frac{1}{2}$$

Hence, $\xi(\cdot,\epsilon)$ is invertible and let its inverse be $\eta(\cdot,\epsilon)$, that is if $\xi(x,\epsilon)=y$, then $\eta(y,\epsilon)=x$ or $\xi(\eta(y,\epsilon),\epsilon)=y$, and by differentiating the latter we get

$$\left.\begin{aligned} \xi_x(\eta(y,\epsilon),\epsilon)\eta_y(y,\epsilon) = 1 \\ \xi_x(\eta(y,\epsilon),\epsilon)\eta_\epsilon(y,\epsilon) + \xi_\epsilon(\eta(y,\epsilon),\epsilon) = 0 \end{aligned}\right\} \implies \begin{cases} \eta_y(y,\epsilon) = \dfrac{1}{1+\frac{\varphi'(\eta(y,\epsilon))}{2\|\varphi'\|_{L^\infty}}\epsilon} \\[2ex] \eta_\epsilon(y,\epsilon) = -\dfrac{\frac{\varphi(\eta(y,\epsilon))}{2\|\varphi'\|_{L^\infty}}}{1+\frac{\varphi'(\eta(y,\epsilon))}{2\|\varphi'\|_{L^\infty}}\epsilon} \end{cases}$$

Now, set $u^\epsilon = \overline{u}(\xi(x,\epsilon))$ and notice that, while u^ϵ is not necessarily a minimizer, it lives in $C^1([a,b])$. And so we have

$$I(u^\epsilon) = \int_a^b f(x,u^\epsilon(x),(u^\epsilon)'(x))\,dx$$

$$= \int_a^b f(x,\overline{u}(\xi(x,\epsilon)),\xi_x(x,\epsilon)\overline{u}'(\xi(x,\epsilon))\,dx \underset{x\equiv\eta(y,\epsilon)}{=} \int_a^b f\left(\eta(y,\epsilon),\overline{u}(y),\frac{\overline{u}'(y)}{\eta_y(y,\epsilon)}\right)\eta_y(y,\epsilon)\,dx$$

and since $I(u^\epsilon) \ge I(\overline{u}) = I(u^0)$ we also take that $\left.\dfrac{d}{d\epsilon}I(u^\epsilon)\right|_{\epsilon=0} = 0$. If we denote, then, by $g(\epsilon)$ the last integrand, we find that

$$g'(0) = g'(\epsilon)|_{\epsilon=0} = \left.\left(\eta_{y\epsilon}f + \left(\eta_\epsilon f_x - \overline{u}'\frac{\eta_{y\epsilon}}{\eta_y^2}f_\xi\right)\eta_y\right)\right|_{\epsilon=0}$$

$$= -\frac{1}{2\left\|\varphi'\right\|_{L^\infty}}\Big(\varphi'(y)\big(f(y,\overline{u}(y),\overline{u}'(y)) - \overline{u}'(y)f_\xi(y,\overline{u}(y),\overline{u}'(y))\big) + \varphi(y)f_x(y,\overline{u}(y),\overline{u}'(y))\Big)$$

which is due to $\eta(y,0) = \mathrm{id}$ (Why?). Thus, we take

$$0 = \frac{d}{d\epsilon} I(u^\epsilon)\Big|_{\epsilon=0} = -\frac{1}{2\|\varphi'\|_{L^\infty}} \int_a^b \Big(\big(f(y,\overline{u},\overline{u}') - \overline{u}' f_\xi(y,\overline{u},\overline{u}') \big) \varphi' + f_x(y,\overline{u},\overline{u}')\varphi \Big)\, dy \quad \forall \varphi \in C_0^\infty$$

and so again, being in one dimension, by Lemma 1.11 we have our result. □

Well, did you spot it? That is, the *"variations of the independent variables"* which is supposed to make here its debut. If you didn't see our star, it is advisable you revise the proof before going any further. Now!

In fact, by *independent variables* we mean x and by *variations* we mean exactly the functions $\xi(\cdot,\epsilon)$ for each ϵ with $|\epsilon| \le 1$ which give us - instead of x - a *"variation"* of x in a small region.

Note that this form of E-L equation seems to be particularly useful in cases where f does not depend explicitly on x, i.e. when $f(x,u,\xi) = f(u,\xi)$. Whenever this happens we can integrate the equation and then we get

$$f(u(x),u'(x)) - u'(x) f_\xi(u(x),u'(x)) = k, \qquad k = \textit{constand}$$

This expression is called the *first integral* of the E-L equation.

2.4 Hamiltonian Formulation

In this section we will study the functional

$$J(u,v) = \int_a^b \big(u'(x)v(x) - H(x,u(x),v(x)) \big)\, dx$$

where H is the Legendre transform of f, namely

$$H(x,u,v) = \sup_{\xi \in \mathbb{R}} \{ v\xi - f(x,u,\xi) \}$$

Sometimes, especially in classical mechanics, f is called the *Lagrangian* whereas H is called the *Hamiltonian*. Now, if we consider the E-L equations of J, we end up with the following system which sometimes goes under the name of *canonical form of the E-L equation*:

$$\begin{cases} u'(x) = H_v(x,u(x),v(x)) \\ v'(x) = -H_u(x,u(x),v(x)) \end{cases} \tag{H}$$

We start our study with a lemma.

Lemma 2.3. *Let $f \in C^2([a,b] \times \mathbb{R} \times \mathbb{R})$ such that*

$$f_{\xi\xi}(x,u,\xi) > 0, \qquad \forall (x,u,\xi) \in [a,b] \times \mathbb{R} \times \mathbb{R} \tag{2.3}$$

$$\lim_{|\xi| \to +\infty} \frac{f(x,u,\xi)}{|\xi|} = +\infty, \qquad \forall (x,u) \in [a,b] \times \mathbb{R} \tag{2.4}$$

and suppose $H(x,u,v) = \sup_{\xi\in\mathbb{R}}\{v\xi - f(x,u,\xi)\}$*. Then* $H \in C^2([a,b]\times\mathbb{R}\times\mathbb{R})$ *and*

$$H_x(x,u,v) = -f_x(x,u,H_v(x,u,v)) \tag{2.5}$$
$$H_u(x,u,v) = -f_u(x,u,H_v(x,u,v)) \tag{2.6}$$
$$H(x,u,v) = vH_v(x,u,v) - f(x,u,H_v(x,u,v)) \tag{2.7}$$
$$v = f_\xi(x,u,\xi) \iff \xi = H_v(x,u,v) \tag{2.8}$$

Remark. *i) The same proof with an extra argument of mathematical deduction can lead us to the conclusion that* $H \in C^k$ *whenever* $f \in C^k$ *with* $k \geq 2$*.*

ii) The condition (2.3) *could be replaced with*

$$\xi \to f(x,u,\xi) \text{ is strictly convex}$$

but then H *would be only* C^1 *ruining thus the beautiful property described in remark* (i)*.*

iii) The hypothesis (2.4) *can also be completely omitted, but then* H *may not be everywhere finite anymore as seen in Theorem 1.37.*

Proof. For starters, observe that the definition of H combined with (2.4) gives that for a fixed point $(x,u)\in[a,b]\times\mathbb{R}$ there exists $\xi = \xi(x,u,v)$ such that

$$H(x,u,v) = v\xi - f(x,u,\xi) \quad \text{and} \quad v = f_\xi(x,u,\xi) \tag{2.9}$$

(the second equality by differentiating the first in regard to ξ). Additionally, we find that H is continuous. Indeed, by its definition as *supremum* and by (2.9) we get $H(x',u',v') \geq v'\xi - f(x',u',\xi)$ and thus

$$H(x,u,v) - H(x',u',v') \leq (v-v')\xi + (f(x',u',\xi) - f(x,u,\xi))$$

Similarly, we can say that there exists $\xi' = \xi'(x',u',v')$ such that $H(x',u',v') = v'\xi' - f(x',u',v')$ and working in the same way we get the respective reverse inequality and hence, thank's to the continuity of f, the continuity of H. Notice, though, that this is only in the variables of x and u.

We need now to prove that $\xi \in C^1([a,b]\times\mathbb{R}\times\mathbb{R})$. (An alternative proof for this can be done using the inverse function theorem.) Towards that goal, we fix a number $R>0$ and then (2.4) ensures the existence of an $M>0$ such that

$$|\xi(x,u,v)| \leq M \qquad \forall x\in[a,b] \text{ and for every } |u|,|v|\leq R$$

At the same time $f_\xi \in C^1$ and so there exists $c>0$ such that

$$|f_\xi(x,u,\xi) - f_\xi(x',u',\xi')| \leq c(|x-x'| + |u-u'| + |\xi-\xi'|) \tag{2.10}$$

for every $x\in[a,b]$, $|u|,|u'|\leq R$ and $|\xi|,|\xi'|\leq M$. Furthermore, (2.3) tells us that there exists a constant $\gamma>0$ so that

$$f_{\xi\xi}(x,u,\xi) \geq \gamma, \qquad \text{for every } x\in[a,b],\ |u|\leq R \text{ and } |\xi|\leq M$$

Hence, for every $x\in[a,b]$, $|u|\leq R$ and $|\xi|,|\xi'|\leq M$ we take

$$|f_\xi(x,u,\xi) - f_\xi(x,u,\xi')| \geq \gamma|\xi-\xi'| \tag{2.11}$$

Recall also that $v = f_\xi(x,u,\xi(x,u,v))$ by definition and similarly we get $v' = f_\xi(x',u',\xi(x',u',v'))$. If we demand $x \in [a,b]$ and $|u|, |u'|, |v|, |v'| \le R$, we can then combine (2.10) and (2.11) to get

$$\gamma|\xi(x,u,v) - \xi(x',u',v')| \le |f_\xi(x,u,\xi(x,u,v)) - f_\xi(x,u,\xi(x',u',v'))|$$

$$= |v - v' + f_\xi(x',u',\xi(x',u',v')) - f_\xi(x,u,\xi(x',u',v'))| \le |v-v'| + c(|x-x'| + |u-u'| + 0)$$

$$\Longrightarrow \gamma|\xi(x,u,v) - \xi(x',u',v')| \le c|x-x'| + c|u-u'| + |v-v'|$$

which clearly yields the continuity of ξ.

Returning once again to the definition of ξ, we have the equality $v = f_\xi(x,u,\xi(x,u,v))$ which we can differentiate in regard to each variable:

$$\begin{cases} f_{\xi x}(x,u,\xi) + f_{\xi\xi}(x,u,\xi)\,\xi_x = 0 \\ f_{\xi u}(x,u,\xi) + f_{\xi\xi}(x,u,\xi)\,\xi_u = 0 \\ \qquad\qquad\qquad f_{\xi\xi}(x,u,\xi)\,\xi_v = 1 \end{cases}$$

Thus $\xi \in C^1([a,b]\times\mathbb{R}\times\mathbb{R})$, since $f \in C^2([a,b]\times\mathbb{R}\times\mathbb{R})$.

The rest of the lemma, that is, the fact that H lives in $C^2([a,b]\times\mathbb{R}\times\mathbb{R})$ as well as the equations (2.5) - (2.8) can be proved directly by differentiating the first equality from (2.9):

$$H(x,u,v) = v\xi(x,u,v) - f(x,u,\xi(x,u,v))$$

while baring in mind that $v = f_\xi(x,u,\xi(x,u,v))$. These calculations are left as an exercise. □

The next theorem illustrates the importance of $J(u,v)$ and the solutions of its equations. The same remarks as above also apply for this theorem.

Theorem 2.4. *Consider we have the same hypotheses with the previous lemma. If the functions u and v live in $C^2([a,b])$ and (u,v) satisfies the following system for every $x \in [a,b]$*

$$\begin{cases} u'(x) = H_v(x,u(x),v(x)) \\ v'(x) = -H_u(x,u(x),v(x)) \end{cases} \tag{H}$$

then u solves the E-L equation

$$\frac{d}{dx} f_\xi\left(x,u(x),u'(x)\right) = f_u\left(x,u(x),u'(x)\right) \tag{E}$$

Conversely, if $u \in C^2([a,b])$ solves the E-L equation and we set

$$v(x) = f_\xi\left(x,u(x),u'(x)\right), \qquad \forall x \in [a,b]$$

then $(u,v) \in C^2([a,b]) \times C^2([a,b])$ is a solution of the above system (H).

Proof.
($\Longrightarrow$) Let (u,v) satisfy (H). Then by (2.8) of the above lemma we have

$$u' = H_v(x,u,v) \iff v = f_\xi\left(x,u,u'\right)$$

and by (2.6)

$$v' = -H_u(x,u,v) = f_u(x,u,H_v)$$

which combined yield the E-L for u.
($\Longleftarrow$) The fact that $v = f_\xi(x,u,u')$ directly implies

$$u' = H_v(x,u,u')$$

from (2.8). We also have

$$v' = \frac{d}{dx}v = \frac{d}{dx}f_\xi(x,\overline{u},\overline{u}') \overset{(E)}{=} f_u(x,\overline{u},\overline{u}') \overset{(2.6)}{=} -H_u(x,u,v)$$

□

2.5 Hamilton-Jacobi Equation

In this section we devote ourselves into solving the *Hamilton-Jacobi equation* namely

$$S_x(x,u) + H(x,u,S_u(x,u)) = 0, \qquad \forall (x,u) \in [a,b]\times\mathbb{R} \tag{2.12}$$

in regard to $S \in C^2([a,b]\times\mathbb{R})$.

Theorem 2.5. *Let $H \in C^1([a,b]\times\mathbb{R}\times\mathbb{R})$, $H = H(x,u,v)$, and suppose that $S \in C^2([a,b]\times\mathbb{R})$, $S = S(x,u)$, is a solution of the H-J equation* (2.12)*. Assume also that there exists $u \in C^1([a,b])$ such that*

$$u'(x) = H(x,u(x),S_u(x,u(x))), \qquad \forall x \in [a,b] \tag{2.13}$$

If $v(x) \equiv S_u(x,u(x))$, then $(u,v) \in C^1([a,b])\times C^1([a,b])$ is a solution of the system

$$\begin{cases} u'(x) = H_v(x,u(x),v(x)) \\ v'(x) = -H_u(x,u(x),v(x)) \end{cases} \tag{H}$$

If we additionally consider a family $S = S(x,u,r)$ with $r \in \mathbb{R}$ which solves (2.13) *for every $(x,u,r) \in [a,b]\times\mathbb{R}\times\mathbb{R}$, then every u satisfying* (2.13) *is also a solution of*

$$\frac{d}{dx}S_r(x,u(x),r) = 0, \qquad \forall (x,r) \in [a,b]\times\mathbb{R} \tag{2.14}$$

Proof. Note that the first equation of the system is trivial due to our hypotheses, so we only need to confirm the second one. Towards that goal we begin by computing v':

$$v'(x) = S_{ux}(x,u(x)) + u'(x)S_{uu}(x,u(x))$$

After that, we differentiate (2.12) with respect to u to take

$$S_{xu} + H_u(x,u,S_u) + S_{uu}H_v(x,u,S_u) = 0$$

which combined with the previous equality, the fact that $S \in C^2([a,b])$ and the definition of v yields

$$v' = S_{uu}(u' - H_v(x,u,S_u)) - H_u(x,u,S_u) = -H_u(x,u,v)$$

And so, (u,v) is indeed a solution of (H).

Now, for the second part of the theorem we start by differentiating the H-J equation (2.12) with respect to r and then for every $(x, u, r) \in [a, b] \times \mathbb{R} \times \mathbb{R}$ we have

$$0 = \frac{d}{dr}\big(S_x(x, u, r) + H(x, u, S_u(x, u, r))\big) = S_{xr}(x, u, r) + S_{ur}(x, u, r)H_u(x, u, S_u(x, u, r))$$

Thus, for $u = u(x)$ satisfying (2.13), the above equality gives that

$$\frac{d}{dx}S_r(x, u(x), r) = S_{xr}(x, u(x), r) + u'(x)H_u(x, u(x), r) = 0$$

□

Corollary 2.6. *If the Hamiltonian, H, does not depend explicitly on x, then every solution $\bar{S}(u, r)$ of the equation*

$$H(u, \bar{S}(u, r)) = r$$

yields a solution of (2.12) *by doing the substitution*

$$S(x, u, r) = \bar{S}(u, r) - rx$$

Theorem 2.7 (Jacobi Theorem)**.** *Let $H \in C^1([a, b] \times \mathbb{R} \times \mathbb{R})$ and suppose $S(\cdot, \cdot, r) \in C^2([a, b] \times \mathbb{R})$ is a solution of the H-J equation* (2.12) *for every $r \in \mathbb{R}$ and it satisfies*

$$S_{ur}(x, u, r) \neq 0, \qquad \forall (x, u, r) \in [a, b] \times \mathbb{R} \times \mathbb{R}$$

If there exists $u = u(x)$ solving (2.14)*, that is*

$$\frac{d}{dx}S_r(x, u(x), r) = 0, \qquad \forall (x, r) \in [a, b] \times \mathbb{R}$$

then u is also a solution of (2.13)*, namely*

$$u'(x) = H_v(x, u(x), S_u(x, u(x), r)), \qquad \forall x \in [a, b]$$

This means, appealing to our previous theorem, that if $v(x) \equiv S_u(x, u(x), r)$, then the system (H) *has $(u, v) \in C^1([a, b]) \times C^1([a, b])$ as a solution.*

Proof. From (2.14) we have that

$$S_{rx}(x, u(x), r) + u'(x)S_u(x, u(x), r) = 0, \qquad \forall (x, r) \in [a, b] \times \mathbb{R}$$

and by differentiating H-J equation (2.12) we get for every $(x, u, r) \in [a, b] \times \mathbb{R} \times \mathbb{R}$

$$0 = \frac{d}{dr}\big(S_x(x, u, r) + H(x, u, S_u(x, u, r))\big) = S_{xr}(x, u, r) + S_{ur}(x, u, r)H_v(x, u, S_u(x, u, r))$$

Combing these two identities and since $S \in C^2([a, b] \times \mathbb{R} \times \mathbb{R})$ and $S_{ur} \neq 0$ we get that

$$u'(x) = H_v(x, u(x), S_u(x, u(x), r)), \qquad \forall (x, r) \in [a, b] \times \mathbb{R}$$

Concerning now the second equation of the system (H), the definition of v and the fact that S solves the H-J equation allow us to calculate v':

$$\begin{aligned} v'(x) &= S_{ux}(x,u(x),r) + u'(x)S_{uu}(x,u(x),r) \\ &= S_{ux}(x,u(x),r) + H_v(x,u(x),S_u(x,u(x),r))S_{uu}(x,u(x),r) \\ &= -H_u(x,u(x),S(x,u(x),r)) + \frac{d}{du}[S_x(x,u,r) + H(x,u,S(x,u,r))]\Big|_{u=u(x)} \\ &= -H_u(x,u(x),v(x)) \end{aligned}$$

□

Chapter 3

Direct Methods

3.1 Introduction

In this chapter we will study under what hypotheses the more general problem

$$\inf\left\{I(u)=\int_{\Omega} f\left(x,u(x),\nabla u(x)\right)\,dx\ :\ u\in u_0+W_0^{1,p}(\Omega)\right\}=m \tag{P}$$

where $\Omega\subseteq\mathbb{R}^n$ is a bounded open with Lipschitz boundary, $u_0\in W^{1,p}(\Omega)$ and $f:\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$, has a solution. It turns out that f generally needs to be convex and coercive.

We start our study with the simpler case of the Dirichlet integral with Lagrangian

$$f\left(x,u,\xi\right)=\frac{1}{2}|\xi|^2$$

and immediately after we pursue further generality. Next we study the E-L equations of our problem attempting to find under what hypothesis on f and u they hold and then we take a glimpse of what happens in the even more general case when $u:\mathbb{R}^n\to\mathbb{R}^m$. Lastly, we inspect the case when the function f fails to be convex.

3.2 Dirichlet Integral

We start with our main theorem:

Theorem 3.1. *Suppose $\Omega\subseteq\mathbb{R}^n$ is open and bounded with a Lipschitz boundary and let $u_0\in W^{1,p}(\Omega)$. Then, the problem*

$$\inf\left\{I(u)=\frac{1}{2}\int_{\Omega}|\nabla u(x)|^2\,dx\ :\ u\in u_0+W_0^{1,p}(\Omega)\right\}=m \tag{P}$$

has a unique solution $\overline{u}\in u_0+W_0^{1,p}(\Omega)$, satisfying the weak form of the E-L equation, that is

$$\int_{\Omega}\langle\nabla\overline{u}(x)\,;\nabla\varphi(x)\rangle\;dx=0,\qquad\forall\varphi\in W_0^{1,2}(\Omega) \tag{3.1}$$

The E-L equation in this form is also known as the weak Laplace equation*.*

Conversely, if $\overline{u}\in u_0+W_0^{1,2}(\Omega)$ satisfies (3.1)*, then it is necessarily a solution of* (P)*.*

Remark. *Observe that should we have extra regularity on $\overline{u}$, i.e. $\overline{u} \in W^{2,2}(\Omega)$, we can integrate* (3.1) *by parts and take*

$$\int_\Omega \Delta\overline{u}(x)\varphi(x)\,dx = 0, \qquad \forall\varphi \in W_0^{1,2}(\Omega)$$

since $\varphi(\partial\Omega) = 0$. Then, the Fundamental Lemma of Calculus of Variations 1.13 tells us that $\overline{u}$ also solves the weak Laplace equation, namely $\Delta\overline{u} = 0$.

By the way, notice that this is our very first existence result in our study. Till now we only have been proposing equivalence results and correlations with the famous named equations. Before we dive into the deep we present a definition and a lemma that we are going to need.

Definition 3.2. *Suppose we have a sequence (u_n) that weakly converges to u. We say that a functional F is* lower semi-continuous *(with respect to sequences) when for every such sequence it holds that $\liminf\limits_{n\to+\infty} F(u_n) \geq F(u)$.*

Lemma 3.3 (Weak lower semi-continuity)**.** *Let Ω be as in the Theorem 3.1. Then, the functional $I(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2\,dx$ is lower semi-continuous.*

Proof. Let $u_u \rightharpoonup u$ in $W^{1,2}$. With some simple calculations we find that

$$\frac{1}{2}|\nabla u_n|^2 = \frac{1}{2}|\nabla u|^2 + \langle \nabla u\,;\nabla u_n - \nabla u\rangle + \frac{1}{2}|\nabla u_n - \nabla u|^2 \geq \frac{1}{2}|\nabla u|^2 + \langle \nabla u\,;\nabla u_n - \nabla u\rangle$$

and by integrating

$$I(u_n) \geq I(u) + \int_\Omega \langle \nabla u\,;\nabla u_n - \nabla u\rangle\,dx \tag{3.2}$$

Now observe that by the definition of the weak convergence we have that $\nabla u \in L^2$ and $\nabla u_n - \nabla u \rightharpoonup 0$ in L^2 which, again by definition, implies

$$\lim_{n\to+\infty} \int_\Omega \langle \nabla u\,;\nabla u_n - \nabla u\rangle\,dx = 0$$

Combining this result with (3.2), we get

$$\liminf_{n\to+\infty} I(u_n) \geq I(u) \tag{3.3}$$

which is what we came here for. □

We say that a sequence (u_n) is a *minimizing sequence* for I when $I(u_n) \to m = \inf\{I(u)\}$. The idea behind the proof of the existence is that we can always have a minimizing sequence in $u_0 + W_0^{1,p}(\Omega)$ which turns out to have some subsequence converging to a function $\overline{u} \in u_0 + W_0^{1,2}(\Omega)$. This endows our space with some sense of *compactness*. With the help of the above lemma we can therefore deduce that $\overline{u}$ has to be a minimizer.

So, without further ado we proceed to the details.

Proof of the theorem. We begin with our much desired result of existence.

Observe that $u_0 \in u_0 + W_0^{1,2}(\Omega)$ and $u_0 \in W^{1,2}$ which gives that

$$0 \leq m \leq I(u_0) < +\infty$$

So, $m \in \mathbb{R}$ and from the definition of *infimum* there exists a minimizing sequence (u_n) with $u_n \in u_0 + W_0^{1,2}$. Namely

$$I(u_n) \to \inf\{I(u)\} = m$$

and thus $I(u_n)$ is bounded, say $I(u_n) \leq M$ for $n \in \mathbb{N}$ large enough. Invoking Poincaré Inequality (Theorem 1.31), there exist constants $c_1,\ c_2 > 0$ such that

$$\sqrt{2I(u_n)} = \|\nabla u_n\|_{L_2} \geq c_1 \|u_n\|_{W^{1,2}} - c_2 \|u_0\|_{W^{1,2}}$$

$$\implies \|u_n\|_{W^{1,2}} \leq \frac{1}{c_1}(c_2 \|u_0\|_{W^{1,2}} + \sqrt{2M}) \equiv c$$

Then, Theorem 1.29 implies that there is a subsequence (u_{n_k}) of (u_n) and an element $\overline{u} \in u_0 + W_0^{1,2}(\Omega)$ such that

$$u_{n_k} \rightharpoonup \overline{u} \text{ in } u_0 + W_0^{1,2}(\Omega)$$

And now, applying the above Lemma 3.3 to (u_{n_k}) we take

$$m = \inf\{I(u)\} = \lim_{k\to+\infty} I(u_{n_k}) \geq \liminf_{k\to+\infty} I(u_{n_k}) \geq I(\overline{u}) \geq m$$

This means that $I(\overline{u}) = m$, hence $\overline{u}$ is a minimizer of (P).

Before everything else, though, it goes without saying that we have to prove the uniqueness of our minimizer. This will be a consequence of the strict convexity of our function $f(\xi) = \frac{1}{2}|\xi|^2$.

Indeed, suppose we have two minimizers, say $\overline{u}, \overline{v} \in u_0 + W_0^{1,2}$ and consider the function $w \equiv \dfrac{\overline{u}+\overline{v}}{2} \in u_0 + W_0^{1,2}(\Omega)$. Then, by convexity, we can infer that w is also a minimizer:

$$|\nabla w|^2 = \left|\nabla(\frac{\overline{u}+\overline{v}}{2})\right|^2 \leq \frac{1}{2}|\nabla\overline{u}|^2 + \frac{1}{2}|\nabla\overline{v}|^2 \tag{3.4}$$

and by dividing with 2 and integrating

$$m \leq I(w) = I(\frac{\overline{u}+\overline{v}}{2}) \leq \frac{I(\overline{u}) + I(\overline{v})}{2} = \frac{m+m}{2} = m \tag{3.5}$$

$$\implies I(w) = m$$

The inequalities in (3.5) also imply that

$$\frac{1}{2}\int_\Omega \frac{1}{2}|\nabla\overline{u}|^2 + \frac{1}{2}|\nabla\overline{v}|^2 - \left|\nabla(\frac{\overline{u}+\overline{v}}{2})\right|^2 dx = \frac{I(\overline{u}) + I(\overline{v})}{2} - I(\frac{\overline{u}+\overline{v}}{2}) = 0$$

However, due to (3.4) the integrand function is non-negative and for its integral to be 0 we need to have

$$\frac{1}{2}|\nabla\overline{u}|^2 + \frac{1}{2}|\nabla\overline{v}|^2 = \left|\nabla(\frac{\overline{u}+\overline{v}}{2})\right|^2$$

almost everywhere in Ω. And because the function $\xi \to \frac{1}{2}|\xi|^2$ is strictly convex the only case this is possible is when $\nabla\overline{u} = \nabla\overline{v}$ almost everywhere in Ω. This in turn yields that $\overline{u} = \overline{v} \in u_0 + W_0^{1,2}(\Omega)$ almost everywhere in Ω, since $\overline{u} = \overline{v} = u_0$ in $\partial\Omega$, and thus the uniqueness of our minimizer.

We have yet to prove that $\overline{u}$ solves the weak Laplace equation (3.1). Let $\epsilon \in \mathbb{R}$ and $\varphi \in W_0^{1,2}(\Omega)$ be arbitrary and notice that $\overline{u} + \epsilon\varphi \in u_0 + W_0^{1,2}$. So, using our familiar trick we have that

$$I(\overline{u}) \leq I(\overline{u} + \epsilon\varphi) = \frac{1}{2}\int_\Omega |\nabla\overline{u} + \epsilon\nabla\varphi|^2\,dx = I(\overline{u}) + \epsilon\int_\Omega \langle\nabla\overline{u}\,;\nabla\varphi\rangle\;dx + \epsilon^2 I(\varphi), \qquad \forall\epsilon \in \mathbb{R},\ \forall\varphi \in W_0^{1,2}$$

So, thank's to Fermat's Theorem we get

$$0 = \frac{d}{d\epsilon}I(\overline{u} + \epsilon\varphi)\bigg|_{\epsilon=0} = \int_\Omega \langle\nabla\overline{u}\,;\nabla\varphi\rangle\;dx + 0, \qquad \forall\varphi \in W_0^{1,2}(\Omega)$$

which is the desired result.

Lastly, we need to prove the converse of the theorem. Suppose that $\overline{u} \in u_0 + W_0^{1,2}$ solves (3.1) and let u be any element in $u_0 + W_0^{1,2}$. We have to show that $I(u) \geq I(\overline{u})$. If $\psi \equiv u - \overline{u}$, then $\psi \in W_0^{1,2}$ and we have

$$I(u) = I(\overline{u} + \psi) = \frac{1}{2}\int_\Omega |\nabla\overline{u} + \nabla\psi|^2\,dx = I(\overline{u}) + I(\psi) + \int_\Omega \langle\nabla\overline{u}\,;\nabla\psi\rangle\;dx \overset{(3.1)}{=} I(\overline{u}) + I(\psi) \geq I(\overline{u})$$

which completes the proof. □

Remark. *The watchful reader may have wondered why doesn't the strict convexity of f give a strict inequality directly in* (3.4) *which could immediately result in the uniqueness of the minimizer. Recall that all the functions considered above are functions in the quotient Sobolev space with equivalence relation "$u = v$ in Ω" whenever "$u = v$ almost everywhere in Ω". The fact that $(u,\xi) \to f(x,u,\xi)$ is convex implies that it is also continuous in the interior of Ω (that is, in the whole Ω since it is an open set) as we already saw at Theorem 1.34. Nevertheless,* (3.4) *holds for every $x \in \overline{\Omega} = \partial\Omega \cup \Omega$ and there may exist $x \in \overline{\Omega}$ for which* (3.4) *holds as an equality regardless of the strict convexity. This is due to two reasons. For one, in $\partial\Omega$ we have no essential restrictions for u, or equivalently for u_0. Secondly, even if we know how u behaves in the whole $\overline{\Omega}$, we have barely any information about its gradient ∇u.*

This remark applies also the proof of the general case discussed in the next section. There, one may find it easier to comprehend the problems described above.

The following corollary is left as an exercise.

Corollary 3.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and $g \in L^2(\Omega)$. Then, the problem*

$$\inf\left\{I(u) = \int_\Omega \big(\frac{1}{2}|\nabla u(x)|^2 - g(x)u(x)\big)\,dx \ :\ u \in W_0^{1,2}(\Omega)\right\} = m \tag{P}$$

admits a unique minimizer, $\overline{u} \in W_0^{1,2}(\Omega)$, for which additionally holds that

$$\int_\Omega \langle\nabla\overline{u}(x)\,;\nabla\varphi(x)\rangle\;dx = \int_\Omega g(x)\varphi(x)\,dx, \qquad \forall\varphi \in W_0^{1,2}(\Omega)$$

3.3 The general case

In this section we are going to see under what hypotheses we have existence of minimizers in the case of any $f \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$. One should take time to compare the following results with Theorem (3.1).

Theorem 3.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and suppose we have a function $f \in C^1(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ having the following properties:*

$$(u,\xi) \to f(x,u,\xi) \text{ is convex} \qquad \forall x \in \overline{\Omega} \tag{C 1}$$

$$\begin{gathered}\text{there exist } p > 1 \text{ and } c_1 > 0, c_3 \in \mathbb{R} \text{ such that} \\ f(x,u,\xi) \geq c_1|\xi|^p + c_3, \qquad \forall (x,u,\xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n\end{gathered} \tag{C 2}$$

$$\begin{gathered}\text{there exists a constant } c \geq 0 \text{ such that} \\ |f_u(x,u,\xi)|, |f_\xi(x,u,\xi)| \leq c(1 + |u|^{p-1} + |\xi|^{p-1}), \qquad \forall (x,u,\xi) \in (\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)\end{gathered} \tag{C 3}$$

If $u_0 \in W^{1,p}(\Omega)$ with $I(u_0) < +\infty$, then there exists a minimizer, $\overline{u} \in u_0 + W_0^{1,p}(\Omega)$, of the problem

$$\inf\left\{ I(u) = \int_\Omega f(x,u(x),\nabla u(x))\, dx \ : \ u \in u_0 + W_0^{1,p}(\Omega) \right\} = m \tag{P}$$

If, additionally, $(u,\xi) \to f(x,u,\xi)$ is strictly convex *for every $x \in \overline{\Omega}$, then the minimizer is unique.*

Before stepping into the proof we need to generalise the previous Lemma 3.3:

Lemma 3.6 (Weak lower semi-continuity)**.** *Let f and Ω be as in the Theorem 3.5. Then, the functional $I(u) = \int_\Omega f(x,u(x),\nabla u(x))\, dx$ is lower semi-continuous (with respect to sequences).*

Note that here we don't have a specific f as in the previous section so we will be needing the information provided by the theorem.

Proof. The convexity of f along with the fact that f is C^1 tells us that

$$f(x,u_n,\nabla u_n) \geq f(x,u,\nabla u) + \big(f_u(x,u,\nabla u)(u_n - u) + \langle f_\xi(x,u,\nabla u)\,; \nabla u_n - \nabla u\rangle\big) \tag{3.6}$$

(cf. Theorem 1.21). Now we would like to integrate (3.6) just as we did before in the proof of the lemma in the Dirichlet case, but we don't know whether the part into the brackets is integrable or not. Fortunately, using (C 3) we can show that each of the addends lives in $L^1(\Omega)$. To begin with, we show that

$$f_u(x,u,\nabla u) \in L^q(\Omega) \quad \text{and} \quad f_\xi(x,u,\nabla u) \in L^q(\Omega;\mathbb{R}^n) \tag{3.7}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Studying the function

$$g_p(a,b) = \frac{(1 + a^{p-1} + b^{p-1})^p}{(1 + a^p + b^p)^{p-1}}$$

with $a, b \geq 0$, we find that it is bounded with $g_p(a,b) \leq 3$ for every $p > 1$ and so

$$(1 + a^{p-1} + b^{p-1})^{\frac{p}{p-1}} \leq 3^{\frac{1}{p-1}}(1 + a^p + b^p)$$

Integrating this last inequality for $a = |u|$ and $b = |\nabla u|$, we get that

$$\int_\Omega (1 + |u|^{p-1} + |\nabla u|^{p-1})^q \, dx \leq 3^{\frac{1}{p-1}} \int_\Omega (1 + |u|^p + |\nabla u|^p) \, dx = 3^{\frac{1}{p-1}} (\mathrm{m}(\Omega) + \|u\|^p_{W^{1,p}})$$

Hypothesis (C 3) and the fact that Ω is bounded imply then

$$\int_\Omega |f_u\,(x, u, \nabla u)\,|^q \leq 3^{\frac{1}{p-1}} (\mathrm{m}(\Omega) + \|u\|^p_{W^{1,p}}) < +\infty$$

thus $f_u\,(x, u, \nabla u) \in L^q(\Omega)$. Similarly, we have that $f_\xi\,(x, u, \nabla u) \in L^q(\Omega; \mathbb{R}^n)$. With the help of Hölder's inequality we have, then, that

$$\|f_u(\cdot, u, \nabla u)(u_n - u)\|_{L^1} \leq \|f_u(\cdot, u, \nabla u)\|_{L^q} \|u_n - u\|_{L^p} < +\infty$$

The same way with f_ξ. We can now integrate (3.6) and end up with

$$I(u_n) \geq I(u) + \int_\Omega f_u\,(x, u, \nabla u)\,(u_n - u)\,dx + \int_\Omega \langle f_\xi(x, u, \nabla u)\,; \nabla u_n - \nabla u\rangle \; dx \tag{3.8}$$

Because $u_n \rightharpoonup u$ in $W^{1,p}$, we have by definition that $u_n \rightharpoonup u$ in L^p and $\nabla u_n \rightharpoonup \nabla u$ in L^p and then again by definition and (3.7) we take

$$\lim_{n\to+\infty} \int_\Omega f_u\,(x, u, \nabla u)\,(u_n - u)\,dx = \lim_{n\to+\infty} \int_\Omega \langle f_\xi(x, u, \nabla u)\,; \nabla u_n - \nabla u\rangle \; dx = 0$$

Finally, (3.8) gives the desired result

$$\liminf_{n\to+\infty} I(u_n) \geq I(u)$$

□

We are ready now to confront our theorem. Note that the ideas behind its proof are exactly the same as in Section 3.2 where we dealing with Dirichlet case.

Proof of the theorem. We begin our proof with the existence result.

Due to our assumptions and (C 2) we have that

$$-\infty < m \leq I(u_0) < +\infty$$

which means that $m = \inf\{I(u)\} \in \mathbb{R}$. By the definition of *infimum* there has to be a minimizing sequence, i.e. a sequence (u_n) that satisfies

$$I(u_n) \to m = \inf\{I(u)\}$$

Thus, $I(u_n)$ is bounded, say $I(u_n) \leq M$, for $n \in \mathbb{N}$ large enough. From (C 2) we, then, have

$$f(x, u_n, \nabla u_n) \geq c_1 |\nabla u_n|^p + c_3 \geq c_1 |\nabla u_n|^p - |c_3|$$

$$\implies M \geq I(u_n) \geq c_1 \|\nabla u_n\|^p_{L^p} - |c_3| \mathrm{m}(\Omega)$$

$$\implies \|\nabla u_n\|_{L^p} \leq c_1^{-\frac{1}{p}}(M + |c_3|\mathrm{m}(\Omega))^{\frac{1}{p}} \equiv c_4$$

And appealing to the Poincaré Inequality there exist $a, b \geq 0$ such that

$$c_4 \geq \|\nabla u_n\|_{L^p} \geq a \|\nabla u_n\|_{W^{1,p}} - b \|\nabla u_0\|_{W^{1,p}}$$

$$\implies \|\nabla u_n\|_{W^{1,p}} \leq \frac{1}{a}(c_4 + b \|\nabla u_0\|_{W^{1,p}}) \equiv \beta$$

Now, we make use of Theorem 1.29 which, since $p > 1$, grants us the existence of a subsequence (u_{n_k}) of (u_n) and a function $\overline{u} \in u_0 + W_0^{1,p}(\Omega)$ so that

$$u_{n_k} \rightharpoonup \overline{u} \text{ in } u_0 + W_0^{1,p}(\Omega)$$

Just like when we had the Dirichlet integral this implies some sort of *compactness* in our space.

Applying Lemma 3.6 to (u_{n_k}) and combining the above compactness result we have that there exists $\overline{u} \in u_0 + W_0^{1,p}$ such that

$$m = \inf\{I(u)\} = \lim_{k\to+\infty} I(u_{n_k}) \geq \liminf_{k\to+\infty} I(u_{n_k}) \geq I(\overline{u}) \geq m$$

$$\implies I(\overline{u}) = m$$

meaning that $\overline{u}$ is a minimizer of (P).

We now have to show the uniqueness of such a minimizer. Suppose that we have two minimizers, say $\overline{u}$ and $\overline{v}$ in $u_0 + W_0^{1,p}(\Omega)$, and set $w \equiv \frac{1}{2}(\overline{u} + \overline{v})$. Note that $w \in u_0 + W_0^{1,p}$ and, since $(u, \xi) \to f(x, u, \xi)$ is convex, we take

$$f(x, w, \nabla w) \leq \frac{f(x, \overline{u}, \nabla\overline{u}) + f(x, \overline{v}, \nabla\overline{v})}{2}$$

$$\implies m \leq I(w) \leq \frac{1}{2}I(\overline{u}) + \frac{1}{2}I(\overline{v}) = \frac{1}{2}m + \frac{1}{2}m = m$$

meaning that w is also a minimizer of (P), i.e. $I(w) = m$. Moreover, it holds

$$\int_\Omega \frac{1}{2}f(x, \overline{u}, \nabla\overline{u}) + \frac{1}{2}f(x, \overline{v}, \nabla\overline{v}) - f(x, w, \nabla w)\, dx = \frac{1}{2}I(\overline{u}) + \frac{1}{2}I(\overline{v}) - I(w) = 0$$

while the integrand is non-negative due to the above inequalities. Hence, we have that

$$\frac{1}{2}f(x, \overline{u}, \nabla\overline{u}) + \frac{1}{2}f(x, \overline{v}, \nabla\overline{v}) = f(x, w, \nabla w) \text{ almost everywhere in } \Omega$$

Strict convexity of f implies, then, that $\overline{u} = \overline{v}$ and $\nabla\overline{u} = \nabla\overline{v}$ almost everywhere in Ω which in turn means that the minimizer is unique, since $\overline{u} = \overline{v}$ almost everywhere in Ω. □

The truth is that the above theorem can be generalized even more without so many restrictions on the function f. The proof, though, of the generalized version we are about to present refrains from the purposes of this study.

Theorem 3.7. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and suppose $f \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ satisfies the following properties:*

$$\xi \to f(x,u,\xi) \text{ is convex} \qquad \forall (x,u) \in \overline{\Omega} \times \mathbb{R} \tag{C}$$

$$\begin{gathered} \text{there exist } p > q \geq 1 \text{ and } c_1 > 0, c_2, c_3 \in \mathbb{R} \text{ such that} \\ f(x,u,\xi) \geq c_1|\xi|^p + c_2|u|^q + c_3, \qquad \forall (x,u,\xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \end{gathered} \tag{$\tilde{\text{C}}$}$$

If $u_0 \in W^{1,p}(\Omega)$ with $I(u_0) < +\infty$, then there exists a minimizer, $\overline{u} \in u_0 + W_0^{1,p}(\Omega)$, of the problem

$$\inf\left\{ I(u) = \int_\Omega f(x,u(x),\nabla u(x))\, dx \;:\; u \in u_0 + W_0^{1,p}(\Omega) \right\} = m \tag{P}$$

If, additionally, $(u,\xi) \to f(x,u,\xi)$ is convex for every $x \in \overline{\Omega}$ and either one $u \to f(x,u,\xi)$ or $\xi \to f(x,u,\xi)$ is strictly convex*, then the minimizer is unique.*

Remark. *One may derive* (C 2) *of Theorem 3.5 from* ($\tilde{\text{C}}$) *by setting $c_2 = 0$. Also, note that these conditions denote that f can be bounded from bellow by some convex function. We will find this pretty useful later at Section 3.6 where we try to "relax" the notion of the minimizer.*

3.4 Euler-Lagrange Equations

In this section we are going to study under what circumstances the minimizer of the problem

$$\inf\left\{ I(u) = \int_\Omega f(x,u(x),\nabla u(x))\, dx \;:\; u \in u_0 + W_0^{1,p}(\Omega) \right\} = m \tag{P}$$

if it exists, satisfies the E-L and the weak E-L equations.

Theorem 3.8. *Let $p \geq 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Suppose we have a function $f \in C^1(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ that satisfies the condition* (C 3) *of Theorem 3.5*

$$\begin{gathered} \text{there exists a constant } c \geq 0 \text{ such that} \\ |f_u(x,u,\xi)|, |f_\xi(x,u,\xi)| \leq c(1 + |u|^{p-1} + |\xi|^{p-1}), \qquad \forall (x,u,\xi) \in (\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n) \end{gathered} \tag{C 3}$$

If $\overline{u} \in u_0 + W_0^{1,p}$, where $u_0 \in W^{1,p}$, is a minimizer of

$$\inf\left\{ I(u) = \int_\Omega f(x,u(x),\nabla u(x))\, dx \;:\; u \in u_0 + W_0^{1,p}(\Omega) \right\} = m \tag{P}$$

then $\overline{u}$ satisfies the weak form of the E-L equation, namely

$$\int_\Omega f_u(x,u,\nabla u)\,\varphi + \langle f_\xi(x,u,\nabla u)\,;\nabla\varphi\rangle\, dx = 0, \qquad \forall \varphi \in W_0^{1,p}(\Omega) \tag{E_w}$$

If f is additionally C^2 and $\overline{u} \in C^2(\overline{\Omega})$, then $\overline{u}$ also solves the E-L equation

$$\sum_{i=1}^n \frac{\partial}{\partial x_i} f_{\xi_i}(x,u,\nabla u) = f_u(x,u,\nabla u) \quad \text{or equivelantly} \quad \operatorname{div}\big(f_\xi(x,u,\nabla u)\big) = f_u(x,u,\nabla u) \tag{E}$$

for every $x \in \overline{\Omega}$.

Conversely, if $\overline{u} \in u_0 + W_0^{1,p}(\Omega)$ is a solution of either (E_w) *or* (E) *and $(u,\xi) \to f(x,u,\xi)$ is convex for every $x \in \overline{\Omega}$, then $\overline{u}$ is a minimizer of* $\inf\{I(u)\} = m$.

Remark. *i) It is obvious that any solution of* (E_w) *is automatically a solution of* (E). *Equation* (E), *however, necessitates the extra hypotheses on the regularity of f and $\overline{u}$ so that is has a meaning.*

ii) Hypothesis (C 3) *is necessary in order for $f_u\varphi$ and $\langle \nabla\varphi\,;f_\xi\rangle$ in the equation* (E_w) *to be in L^1.*

iii) One can derive an even weaker form of the E-L equation whenever the test functions φ live in $C_0^\infty(\Omega)$ instead of the Sobolev space $W_0^{1,p}(\Omega)$. In this context we can weaken (C 3) *to:*

$$\begin{gathered}\textit{there exists a constant } c \geq 0 \textit{ such that}\\ |f_u\,(x,u,\xi)\,|, |f_\xi\,(x,u,\xi)\,| \leq c(1+|u|^p+|\xi|^p), \qquad \forall\,(x,u,\xi) \in (\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^n)\end{gathered} \tag{C' 3}$$

Proof.
($\Longrightarrow$) For starters, we observe

$$f\,(x,u,\xi) = f(x,0,0) + \int_0^1 \frac{d}{dt} f(x,tu,t\xi)\,dt, \qquad \forall\,(x,u,\xi)\in\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^n$$

$$\Longrightarrow\; f\,(x,u,\xi) = f(x,0,0) + \int_0^1 uf_u(x,tu,t\xi) + \xi f_\xi(x,tu,t\xi)\,dt$$

$$\Longrightarrow\; |f\,(x,u,\xi)\,| \leq |f(x,0,0)| + \int_0^1 |u||f_u(x,tu,t\xi)| + |\xi||f_\xi(x,tu,t\xi)|\,dt, \qquad \forall\,(x,u,\xi)\in\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^n$$

which through (C 3) and because f is C^1 gives that

$$|f\,(x,u,\xi)\,| \leq M + c(|u|+|\xi|)\int_0^1 (1+|tu|^{p-1}+|t\xi|^{p-1})dt = M + c(|u|+|\xi|)(1+\frac{1}{p}|u|^{p-1}+\frac{1}{p}|\xi|^{p-1})$$

where $M = \max_{x\in\overline{\Omega}}\{f(x,0,0)\}$. With the help of Young's Inequality $|uv| \leq \frac{1}{p}|u|^p + \frac{1}{q}|v|^q$, where $\frac{1}{p}+\frac{1}{q}=1$, we can then derive that

$$|f\,(x,u,\xi)\,| \leq b(1+|u|^p+|\xi|^p), \qquad \forall\,(x,u,\xi)\in\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^n \tag{3.9}$$

for some constant b. Hence, $I(u) < +\infty$ for every $u \in u_0 + W^{1,p}(\Omega)$, since u and ∇u live in L^p.

Now, if φ is any function in $W_0^{1,p}(\Omega)$, we want to use Fermat's Theorem for $I(u)$, since $I(\overline{u}+\epsilon\varphi) \geq I(\overline{u})$, and simply take

$$\int_\Omega \big(f_u\,(x,\overline{u}(x),\nabla\overline{u}(x))\,\varphi(x) + \langle f_\xi\,(x,\overline{u}(x),\nabla\overline{u}(x))\;;\nabla\varphi(x)\rangle\,\big)\,dx = \frac{d}{d\epsilon}\,I(\overline{u}+\epsilon\varphi)|_{\epsilon=0} = 0 \tag{3.10}$$

However, the first equality of the above relation has yet to be established. Even though we do know that the function $f(x, u+\epsilon\varphi, \nabla u + \epsilon\nabla\varphi)$ is differentiable with respect to ϵ (because f is C^1), it is not trivial that its derivative is integrable (its integral may not be finite). So, if we set

$$h(x,\epsilon) \equiv f(x, u(x)+\epsilon\varphi(x), \nabla u(x)+\epsilon\nabla\varphi(x))$$

we need to show that $h_\epsilon(x,0)$ is integrable.

For this purpose we are going to use Lebesgue's Dominated Convergence Theorem for the function $\dfrac{h(x,\epsilon)-h(x,0)}{\epsilon}$, while $\epsilon \to 0$. From (3.9) we can easily see that for $\epsilon > 0$ small enough

$$\frac{h(x,\epsilon)-h(x,0)}{\epsilon} \in L^1(\Omega)$$

and we already know that

$$\frac{h(x,\epsilon)-h(x,0)}{\epsilon} \to h_\epsilon(x,0) \text{ almost everywhere in } \Omega$$

Moreover, for every ϵ near 0 hypothesis (C 3) gives us that there exists some constant $\kappa > 0$ so that

$$\left|\frac{h(x,\epsilon)-h(x,0)}{\epsilon}\right| \leq \kappa(1+|u|^p+|\nabla u|^p+|\varphi|^p+|\nabla\varphi|^p) \equiv H(x) \tag{3.11}$$

and $H(x)$ is, of course, in $L^1(\Omega)$. To achieve this inequality, we have to somehow compare these two functions. The only way to do this is through the Mean Value Theorem which tells us that there exists some $t \in (-|\epsilon|, |\epsilon|)$ such that

$$\frac{h(x,\epsilon)-h(x,0)}{\epsilon} = h_\epsilon(x,t) = \varphi f_u(x,u+t\varphi,\nabla u+t\nabla\varphi) + \langle \nabla\varphi\,;f_\xi(x,u+t\varphi,\nabla u+t\nabla\varphi)\rangle$$

$$\begin{aligned} \overset{(\text{C } 3)}{\Longrightarrow}\ |h_\epsilon(x,t)| &\leq c(|\varphi|+|\nabla\varphi|)(1+|u+t\varphi|^{p-1}+|\nabla u+t\nabla\varphi|^{p-1}) \\ &\leq c(|\varphi|+|\nabla\varphi|)\big(1+(|u|+|\varphi|)^{p-1}+(|\nabla u|+|\nabla\varphi|)^{p-1}\big) \end{aligned}$$

for ϵ (and thus t) sufficiently small. Now, studying the function

$$g_p(a,b) = \frac{(a+b)^{p-1}}{a^{p-1}+b^{p-1}}$$

for $a, b \geq 0$ as we did back at Lemma 3.6 we find that it is bounded with $g_p(a,b) \leq 2^{p-2}$ for every $p > 1$. Combined with the above inequality, this tells us that

$$|h_\epsilon(x,t)| \leq \overline{b}(|\varphi|+|\nabla\varphi|)(1+|u|^{p-1}+|\varphi|^{p-1}+|\nabla u|^{p-1}+|\nabla\varphi|^{p-1})$$

for some constant $\overline{b}$. In turn, similar manoeuvres with Young's Inequality as those earlier in this proof lead us to the desired result (3.11).

Therefore, from Lebesgue's Dominated Convergence Theorem we take that $h_\epsilon(x,0)$ is integrable and

$$\int_\Omega \frac{h(x,\epsilon)-h(x,0)}{\epsilon}\,dx \to \int_\Omega h_\epsilon(x,0)$$

$$\Longrightarrow \lim_{\epsilon\to 0}\frac{I(u+\epsilon\varphi)-I(u)}{\epsilon} = \int_\Omega \big(\varphi(x) f_u\left(x,u(x),\nabla u(x)\right) + \langle \nabla\varphi(x)\,; f_\xi\left(x,u(x),\nabla u(x)\right)\rangle\big)\,dx$$

which for $u = \overline{u}$ yields exactly (3.10) and so $\overline{u}$ is a solution of the weak E-L equation, (E_w).

In order to obtain (E) we need to integrate the second addend in (3.10) by parts. Thankfully, Corollary 1.26 enables us to do so and thus we take

$$\int_\Omega \Big(f_u\left(x,\overline{u}(x),\nabla\overline{u}(x)\right) - \operatorname{div}\big(f_\xi\left(x,\overline{u}(x),\nabla\overline{u}(x)\right)\big)\Big)\varphi(x)\,dx = 0 \qquad \forall\varphi \in W_0^{1,p}(\Omega)$$

The Fundamental Lemma of the Calculus of Variations (1.13), then, directly yields the desired result.

($\Longleftarrow$) We can suppose that $\overline{u} \in u_0 + W_0^{1,p}(\Omega)$ is a solution of (E$_w$), since every solution of (E) is also a solution of (E$_w$). Then, from the convexity of $(u,\xi) \to f(x,u,\xi)$ we have that for every $u \in u_0 + W_0^{1,p}$ it holds

$$f(x,u,\nabla u) \geq f(x,\overline{u},\nabla\overline{u}) + f_u(x,\overline{u},\nabla\overline{u})(u-\overline{u}) + \langle f_\xi(x,\overline{u},\nabla\overline{u})\,;\nabla u - \nabla\overline{u}\rangle\,, \qquad \forall x \in \Omega$$

Integrating this inequality and thank's to $\overline{u}$ being a solution of (E$_w$) we get that

$$I(u) \geq I(\overline{u}) + 0$$

for every $u \in u_0 + W_0^{1,p}(\Omega)$, which means $\overline{u}$ is a minimizer of $\inf\{I(u)\} = m$. □

3.5 The vectorial case

In this section we are going to briefly present the generalization of what we have already seen in the case when $f : \Omega \times \mathbb{R}^2 \times \mathbb{R}^{2\times 2} \to \mathbb{R}$. In other words we are searching for the minimizers of $\inf\{I(u)\} = m$ when $u \in u_0 + W_0^{1,p}(\Omega;\mathbb{R}^2)$, with $u_0 \in W^{1,p}(\Omega;\mathbb{R}^2)$.

Theorem 3.9. *Let $n = m = 2$ and suppose $\Omega \subset \mathbb{R}^2$ is a bounded open set with Lipschitz boundary. Consider the continuous function $f : \overline{\Omega} \times \mathbb{R}^2 \times \mathbb{R}^{2\times 2} \to \mathbb{R}$ and suppose there exist another continuous function $F : \overline{\Omega} \times \mathbb{R}^2 \times \mathbb{R}^{2\times 2} \times \mathbb{R} \to \mathbb{R}$ for which we have*

$$\begin{aligned} f(x,u,\xi) &= F(x,u,\xi,\det\xi), && \forall\,(x,u,\xi) \in \overline{\Omega} \times \mathbb{R}^2 \times \mathbb{R}^{2\times 2} \\ (\xi,\delta) &\to F(x,u,\xi,\delta) \ \textit{ is convex} && \forall (x,u) \in \overline{\Omega} \times \mathbb{R} \\ &\textit{there exists } p > \max\{q,2\} \textit{ and } c_1 > 0, c_2, c_3 \in \mathbb{R} \textit{ such that} \\ F(x,u,\xi,\delta) &\geq c_1|\xi|^p + c_2|u|^q + c_3, && \forall (x,u,\xi,\delta) \in \overline{\Omega} \times \mathbb{R}^2 \times \mathbb{R}^{2\times 2} \times \mathbb{R} \end{aligned}$$

If $u_0 \in W^{1,p}(\Omega;\mathbb{R}^2)$ is such that $I(u_0) < +\infty$, then

$$\inf\left\{ I(u) = \int_\Omega f(x,u(x),\nabla u(x))\,dx \ : \ u \in u_0 + W_0^{1,p}(\Omega;\mathbb{R}^2) \right\} = m$$

admits a minimizer.

Remark. *The proof of this theorem makes use of Lemma 1.30 we encountered during our study of weak convergence at preliminaries.*

The generalization of this theorem gets rather complicated. Remember that if A is a $n \times m$ matrix, then by $\operatorname{adj}_r(A)$ we denote the matrix consisting of all the $r \times r$ cofactors of A provided that $n, m \geq r$. The function f would then be of the form $f(x,u,\xi) = F(x,u,\xi,\operatorname{adj}_2\xi,\ldots,\operatorname{adj}_k\xi)$, where $k = \min\{n,m\}$, and we would need the function F to be convex for every fixed point (x,u). Under some extra conditions on F we have that the functional I admits a minimizer. This case was studied by Charles B. Morrey who stated the necessary hypotheses to work with and proved the general theorem in 1952.

Unfortunately, the presentation of these results refrains from the purpose of this study.

3.6 Relaxation Theory

Having seen all these theorems so far, one may wonder: Is there a way to somehow *force* the existence of a minimizer for our familiar problem (P)? The answer is surprisingly *yes*, however, only in the case when we just miss the convexity of the function f. For this purpose we have to use the help of the *convex envelop* of a function that we saw in Section 1.5. Just like the previous section, though, we are not going to see any proofs here.

Theorem 3.10. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and suppose we have a function $f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $f = f(x, u, \xi)$, which is uniformly continuous in $\mathbb{R}^n$ with respect to ξ with convex envelop $f^{\star\star}$ again with respect to ξ. Assume also that there exist $c \in \mathbb{R}$ and $p > 1$ such that*

$$0 \leq f(x, u, \xi) \leq c(1 + |u|^p + |\xi|^p), \qquad \forall (x, u, \xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$$

If $u_0 \in W^{1,p}(\Omega)$ and

$$\inf \left\{ \overline{I}(u) = \int_\Omega f^{\star\star}(x, u(x), \nabla u(x)) \, dx \ : \ u \in u_0 + W_0^{1,p}(\Omega) \right\} = \overline{m} \tag{$\bar{\text{P}}$}$$

then the following hold:

$$\overline{m} = m = \inf\{I(u)\}$$
$$\textit{for every } u \in u_0 + W_0^{1,p}(\Omega) \textit{ there exists } (u_n) \textit{ with } u_n \in u_0 + W_0^{1,p}(\Omega) \textit{ so that}$$
$$u_n \rightharpoonup u \textit{ in } W^{1,p} \textit{ and } I(u_n) \rightharpoonup \overline{I}(u), \textit{ while } n \to +\infty$$

If, additionally, there exist $c_1 > 0, c_3 \in \mathbb{R}$ such that

$$f(x, u, \xi) \geq c_1|\xi|^p + c_3, \qquad \forall (x, u, \xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \tag{$\tilde{\text{C}}$}$$

then ($\bar{\text{P}}$) *admits a minimizer $\overline{u} \in u_0 + W_0^{1,p}(\Omega)$.*

Remark. *Observe that if f satisfies* ($\tilde{\text{C}}$)*, then* ($\tilde{\text{C}}$) *also holds for $f^{\star\star}$. This happens because the function $g(\xi) \equiv c_1|\xi|^p + c_3$ is also convex and, since $f^{\star\star}$ is the convex envelop of f, we have that $f(x, u, \xi) \geq f^{\star\star}(x, u, \xi) \geq g(\xi)$ for every convex function g. By Theorem 3.5 we then immediately get the existence of a minimizer for* $\inf\{\overline{I}(u)\} = \overline{m}$.

Thankfully, we are lucky enough to have the bidual of a function defined under no restrictions on the function itself. Theorem 3.10 enables us now to define a *"generalized"* minimizer of (P) whenever the given f fails to provide us with one - under some conditions, of course.

Chapter 4

Regularity

4.1 Introduction

This chapter, as its title suggests, is devoted to finding the connection between the regularity of $f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and the regularity of a minimizer $\overline{u}$ (whenever it exists) of the problem

$$\inf \left\{ I(u) = \int_\Omega f\left(x, u(x), \nabla u(x)\right)\, dx \ : \ u \in u_0 + W_0^{1,p}(\Omega) \right\} = m$$

where Ω is a bounded open set with Lipschitz boundary and $u_0 \in W^{1,p}(\Omega)$. Working in the same scheme, we begin our study from the case when $n = 1$ and after that we examine the Dirichlet integral before going into further generalization.

4.2 The one dimensional case

Before we begin our study of the minimizers we need to ensure we do have at least one minimizer. Obviously, there is no point study the minimizer if there is no minimizer in the first place. Thankfully, after going through Chapter 3, we are now able to provide a suitable context for this purpose.

Let $f \in C([a, b] \times \mathbb{R} \times \mathbb{R})$ and consider the following hypotheses:

$$\xi \to f\left(x, u, \xi\right) \text{ is convex} \qquad \forall (x, u) \in [a, b] \times \mathbb{R} \tag{C}$$

$$\begin{gathered} \text{there exist } p > q \geq 1 \text{ and } c_1 > 0, c_2, c_3 \in \mathbb{R} \text{ such that} \\ f\left(x, u, \xi\right) \geq c_1|\xi|^p + c_2|u|^q + c_3, \qquad \forall\left(x, u, \xi\right) \in [a, b] \times \mathbb{R} \times \mathbb{R} \end{gathered} \tag{$\tilde{\text{C}}$}$$

Then, if $X = \{u \in W^{1,p}(a, b) \ : \ u(a) = \alpha,\ u(b) = \beta\}$, the problem

$$\inf_{u \in X} \left\{ I(u) = \int_a^b f\left(x, u(x), u'(x)\right)\, dx \right\} = m \tag{P}$$

admits a minimizer, $\overline{u} \in X$, thank's to Theorem 3.7. If, additionally, f lives in $C^1([a, b] \times \mathbb{R} \times \mathbb{R})$ and satisfies (C 3) of Theorem 3.5, namely

$$\begin{gathered} \text{there exists a constant } c = c(R) \text{ such that} \\ \left|f_u\left(x, u, \xi\right)\right|, \left|f_\xi\left(x, u, \xi\right)\right| \leq c(1 + |\xi|^{p-1}), \qquad \forall\left(x, u, \xi\right) \in ([a, b] \times [-R, R] \times \mathbb{R}) \end{gathered} \tag{C 3}$$

then the minimizer $\overline{u}$ satisfies the weak E-L equation

$$\int_a^b f_u(x,u,u')\,\varphi + f_\xi(x,u,u')\,\varphi'\,dx = 0, \qquad \forall\varphi \in W_0^{1,p}(a,b) \tag{E$_w$}$$

Furthermore, if we wish for a weaker form of the E-L equation, we can loose our restriction on the test functions and let them live in $C_0^\infty(a,b)$ instead of $W_0^{1,p}(a,b)$. This allows us to weaken (C 3) even more to (C' 3), that is

$$\begin{gathered}\textit{there exists a constant } c = c(R) \textit{ such that}\\ |f_u(x,u,\xi)|, |f_\xi(x,u,\xi)| \le c(1+|\xi|^p), \qquad \forall\,(x,u,\xi) \in ([a,b]\times[-R,R]\times\mathbb{R})\end{gathered} \tag{C' 3}$$

as we saw at the remark of Section 3.4.

You must have noticed that (C 3) and (C' 3) are not exactly as we left them. Working in only one dimension in the closed interval $[a,b]$, allows us to assume without loss of generality that u is only in $[-R,R]$ when trying to impose a boundary on the derivatives of f.

From now on we need to bear in mind all the above hypotheses and conditions in order to proceed. For starters, we will see an elementary and rather mild result.

Proposition 4.1. *Suppose we have a function $g \in C^\infty([a,b]\times\mathbb{R})$ for which there exist $2 > q \ge 1$ and constants $c_2, c_3 \in \mathbb{R}$ such that*

$$g(x,u) \ge c_2|u|^q + c_3, \qquad \forall(x,u) \in [a,b]\times\mathbb{R}$$

and consider the function

$$f(x,u,\xi) = \frac{1}{2}\xi^2 + g(x,u)$$

Then, there exists a minimizer $\overline{u}$ of (P) *which is $C^\infty([a,b])$. If, additionally, the function $u \to g(x,u)$ happens to be convex for every $x \in [a,b]$, then the minimizer is unique.*

Remark. *i) Note that the property g satisfies is exactly the hypothesis $(\tilde{\mathrm{C}})$ with $p = 2$ and $c_1 = 0$. Unfortunately, we cannot extract any information about g, but this property allows f to satisfy $(\tilde{\mathrm{C}})$, which is exactly what we need.*

ii) The following proof will show that in fact $\overline{u} \in C^{k+1}$ whenever $g \in C^k$ with $k \ge 1$.

Proof. From Theorem 3.7 we find that $p = 2$ and directly take the existence and the uniqueness (provided $u \to g(x,u)$ is convex) of a minimizer $\overline{u} \in X = \{u \in W^{1,2}(a,b) \;:\; u(a) = \alpha,\; u(b) = \beta\}$. Now, since $f \in C^1$, from Theorem 3.8 $\overline{u}$ has to solve the weak E-L equation

$$\int_a^b (f_u(x,u,u')\,\varphi + f_\xi(x,u,u')\,\varphi')\,dx = 0, \qquad \forall\varphi \in C_0^\infty(a,b) \tag{E$_w$}$$

$$\Longrightarrow \int_a^b \overline{u}'(x)\varphi'(x)\,dx = -\int_a^b g_u(x,\overline{u}(x))\varphi(x)\,dx \qquad \forall\varphi \in C_0^\infty(a,b) \tag{4.1}$$

($\overline{u}'$ being only the weak derivative of $\overline{u}$). Note also that $\overline{u} \in W^{1,2}$ which implies $\overline{u} \in L^\infty$. Hence, $x \to g_u(x,\overline{u}(x)) \in L^2$ and we have

$$\left|\int_a^b \overline{u}'\varphi'\,dx\right| \le \int_a^b |g_u(x,\overline{u})\varphi|\,dx \underset{\text{C-S}}{\le} \|g_u(x,\overline{u})\|_{L^2}\,\|\varphi\|_{L^2} \qquad \forall\varphi \in C_0^\infty(a,b)$$

From Theorem 1.21 this inequality tells us that $\overline{u}' \in W^{1,2}$ and so $\overline{u} \in W^{2,2}$. This allows us to integrate (4.1) by parts and the *Fundamental Lemma of the Calculus of Variations* 1.13 gives us, then, that

$$\overline{u}''(x) = g_u(x, \overline{u}(x)) \text{ almost everywhere in } (a, b) \tag{4.2}$$

(We could also have made use of Lemma 1.11 to take (4.2).) Now, from the *Sobolev Embedding Theorem* 1.24 and because $\overline{u} \in W^{2,2}$ we deduce that $\overline{u} \in C^1([a, b])$. Thus, the function $x \to g_u(x, \overline{u}(x))$ is also C^1, since $g \in C^\infty$, and (4.2) tells us that $\overline{u}'' \in C^1 \implies \overline{u} \in C^3$. But then, the function $x \to g_u(x, \overline{u}(x))$ has to also be C^3 meaning that $\overline{u} \in C^5$ and so on. The fact that g is C^∞ implies, then, that $\overline{u}$ is C^∞, too (See the definition of the C^∞ functions, Definition 1.3.). □

It should be obvious that this result is not very satisfactory, however, it helps us illustrate our goals. We immediately seek for further generalization.

Theorem 4.2. *Let $f \in C^\infty([a, b] \times \mathbb{R} \times \mathbb{R})$ satisfy the hypotheses* ($\tilde{\text{C}}$), (C' 3) *and also*

$$f_{\xi\xi}(x, u, \xi) > 0, \qquad \forall (x, u, \xi) \in [a, b] \times \mathbb{R} \times \mathbb{R} \tag{C' 1}$$

Then any minimizer of (P) *lives in $C^\infty([a, b])$.*

Remark. *i) First of all, notice that here we do have a generalization of Proposition 4.1.*

ii) Condition (C' 1) *naturally seems to be quite restrictive. The reason behind this is that we make use of Lemma 2.3 for which we need to have this extra property.*

iii) Just like in proof of 4.1 we will see in the proof that the minimizer is C^k whenever $f \in C^k$, $k \geq 1$.

Before can see the details, though, we need to state another lemma we need.

Lemma 4.3. *If a function $f \in C^1([a, b] \times \mathbb{R} \times \mathbb{R})$ satisfies* (C), ($\tilde{\text{C}}$) *and* (C' 3)*, then any minimizer $\overline{u} \in W^{1,p}(a, b)$ of* (P) *with $p > 1$ lives in fact in $W^{1,\infty}(a, b)$. Moreover, the E-L equation holds for almost every $x \in (a, b)$:*

$$\frac{d}{dx} f_\xi(x, \overline{u}(x), \overline{u}'(x)) = f_u(x, \overline{u}(x), \overline{u}'(x)) \text{ almost everywhere in } (a, b)$$

Proof. We already know that $\overline{u}$ solves the weak E-L equation and so

$$\int_a^b \left(f_u(x, \overline{u}, \overline{u}')\,\varphi + f_\xi(x, \overline{u}, \overline{u}')\,\varphi'\right) dx = 0, \qquad \forall \varphi \in C_0^\infty(a, b) \tag{E$_w$}$$

Additionally, due to (C' 3) and $\overline{u}$ being in $W^{1,p}$ we have that $f_\xi(x, \overline{u}, \overline{u}'), f_u(x, \overline{u}, \overline{u}') \in L^1$. Thus, from Lemma 1.11 we have that $\overline{u}$ solves E-L equation for almost every $x \in (a, b)$.

Now, we need to show that $\overline{u} \in W^{1,\infty}$ or equivalently that $\overline{u} \in L^\infty$ and $\overline{u}' \in L^\infty$. The first one is trivial due to the Sobolev Embedding Theorem 1.24 so we only need to prove the second. At the same time because $f_\xi(x, \overline{u}, \overline{u}') \in W^{1,1}(a, b)$ it also lives in $C([a, b])$ and thus it is bounded, too. Say, then, $|f_\xi(x, \overline{u}, \overline{u}')| \leq M$ and $|\overline{u}(x)| \leq R$ in $[a, b]$ for $M, R \in \mathbb{R}$. We initially have from (C) that

$$f(x, u, 0) \geq f(x, u, \xi) + (0 - \xi) f_\xi(x, u, \xi)$$

$$\implies \xi f_\xi(x,u,\xi) \geq f(x,u,\xi) - f(x,u,0), \qquad \forall (x,u,\xi) \in [a,b] \times [-R,R] \times \mathbb{R}$$

and using ($\tilde{\mathrm{C}}$) we get the existence of constants $\gamma_1, \gamma_2 \in \mathbb{R}$ (that depend on R and M) such that

$$\xi f_\xi(x,u,\xi) \geq \gamma_1|\xi|^p + \gamma_2 \implies \gamma_1|\overline{u}'|^p + \gamma_2 \leq \overline{u}' f_\xi(x,\overline{u},\overline{u}')$$

$$\implies \gamma_1|\overline{u}'|^p + \gamma_2 \leq |\overline{u}'||f_\xi(x,\overline{u},\overline{u}')| \leq M|\overline{u}'|, \text{ almost everywhere in } (a,b) \tag{4.3}$$

With the help of Young's Inequality $|uv| \leq \frac{1}{p}|u|^p + \frac{1}{q}|v|^q$, where $\frac{1}{p} + \frac{1}{q} = 1$, for $v = \epsilon^{-1}$ small enough we can infer from (4.3) that $\overline{u}'$ is bounded completing thus the proof of the lemma. □

Proof of the theorem. We already know that the function $\overline{v}(x) \equiv f_\xi(x,\overline{u}(x),\overline{u}'(x))$, where $\overline{u}$ is a minimizer of (P), is in $W^{1,1}(a,b)$ and thus continuous, thank's to the previous Lemma 4.3. Now, we consider the function

$$H(x,u,v) = \sup_{\xi\in\mathbb{R}}\{v\xi - f(x,u,\xi)\}$$

which, from Lemma 2.3 and because $f \in C^\infty$, lives in $C^\infty([a,b] \times \mathbb{R} \times \mathbb{R})$. The same lemma gives us that

$$\overline{v}(x) = f_\xi(x,\overline{u}(x),\overline{u}'(x)) \iff \overline{u}'(x) = H_v(x,\overline{u}(x),\overline{v}(x))$$

and, since $\overline{v}$, $\overline{u}$ and H_v are continuous, $\overline{u}'$ is also continuous thus $\overline{u}$ is C^1. This ensures the continuity of $x \to f_u(x,\overline{u}(x),\overline{u}'(x))$ and by differentiating $\overline{v}(x)$:

$$\frac{d}{dx}\overline{v}(x) = \frac{d}{dx} f_\xi(x,\overline{u}(x),\overline{u}'(x)) \overset{\text{Lemma 4.3}}{=} f_u(x,\overline{u}(x),\overline{u}'(x)) \text{ almost everywhere in } (a,b)$$

we get that it lives in $C^1([a,b])$. Because $H \in C^\infty$, the associated Hamiltonian system

$$\begin{cases} \overline{u}'(x) = H_v(x,\overline{u}(x),\overline{v}(x)) \\ \overline{v}'(x) = -H_u(x,\overline{u}(x),\overline{v}(x)) \end{cases} \tag{H}$$

allows us, then, to begin an iteration process: $\overline{u}$ and $\overline{v}$ are C^1 thus from (H) $\overline{u}$ and $\overline{v}$ are C^2 which means that they are in turn C^3 and... and... Eventually, C^∞. □

The above result has an alternative yet rather indirect proof. In particular, one can show that $\overline{u} \in W^{2,\infty}(a,b)$ using Theorem 1.21 and then an iteration process can be done through the following form of the E-L equation

$$f_u(x,u,u') = \frac{d}{dx} f_\xi(x,u,u') = u'' f_{\xi\xi}(x,u,u') + u' f_{\xi u}(x,u,u') + f_{\xi x}(x,u,u')$$

This is left as an exercise to the reader.

We shall see now a theorem resembling Proposition 4.1 where hypothesis (C' 1) is redundant.

Theorem 4.4. *Suppose we have a function $g \in C^1([a,b] \times \mathbb{R})$ which satisfies ($\tilde{\mathrm{C}}$) (with $c_1 = 0$), i.e.*

$$\begin{array}{c} \textit{there exist } q \geq 1 \textit{ and } c_2, c_3 \in \mathbb{R} \textit{ such that} \\ g(x,u) \geq c_2|u|^q + c_3, \qquad \forall (x,u) \in [a,b] \times \mathbb{R} \end{array} \tag{$\tilde{\mathrm{C}}$}$$

and consider the function $f \in C^1([a,b]\times\mathbb{R}\times\mathbb{R})$ given by the formula

$$f\,(x,u,\xi) = \frac{1}{p}|\xi|^p + g(x,u)$$

where $p > q \geq 1$. There exists, then, a minimizer $\overline{u} \in C^1([a,b])$ of (P) with $|\overline{u}'|^{p-2}\overline{u}' \in C^1([a,b])$ that also solves almost everywhere the E-L equation, that is

$$\frac{d}{dx}\left(|\overline{u}'(x)|^{p-2}\overline{u}'(x)\right) = \frac{d}{dx}f_\xi\,(x,\overline{u}(x),\overline{u}'(x)) = g_u(x,\overline{u}(x)) \text{ almost everywhere in } [a,b] \tag{4.4}$$

If it additionally happens to be $2 \geq p > 1$, then $\overline{u} \in C^2([a,b])$. The convexity of the function $u \to g(x,u)$, implies as usual the uniqueness of the minimizer.

Proof. Theorem 3.7 guaranties the existence (and the uniqueness whenever we have the convexity) of a minimizer $\overline{u} \in W^{1,p}$ and through Lemma 4.3 we have that $\overline{u} \in W^{1,\infty}$ and that it satisfies the E-L equation almost everywhere. So (4.4) holds. Then, since $\overline{u}$ is continuous, $g_u(x,\overline{u}(x))$ is also continuous, hence $|\overline{u}'|^{p-2}\overline{u}' \in C^1$. Now, if we set $v \equiv |\overline{u}'|^{p-2}\overline{u}'$, we can see that

$$\overline{u}' = |v|^{\frac{2-p}{p-1}}v$$

while v is C^1. Observe that the function $h(\theta) = |\theta|^{\frac{2-p}{p-1}}\theta$ lives in C^1 whenever $\frac{2-p}{p-1} \geq 0 \iff 2 \geq p > 1$ (while it is continuous for every $p > 1$). Since $v \in C^1$, the same property holds for $\overline{u}'(x) = h(v(x))$ yielding thus the desired result. ☐

4.3 The Dirichlet Integral

Suppose we have a bounded open set with Lipschitz boundary $\Omega \subset \mathbb{R}^n$ and a function $u_0 \in W^{1,2}(\Omega)$. Consider our usual problem with the Dirichlet function

$$\inf\left\{I(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2\,dx \;:\; u \in u_0 + W_0^{1,2}(\Omega)\right\} \tag{P}$$

We already know that (P) admits a minimizer $\overline{u} \in u_0 + W_0^{1,2}(\Omega)$ which additionally satisfies the weak Laplace equation

$$\int_\Omega \langle \nabla u(x)\,;\nabla\varphi(x)\rangle\;dx, \qquad \forall\varphi \in W_0^{1,2}(\Omega) \tag{E$_w$}$$

Our purpose is to show that this minimizer lives in $C^\infty(\Omega)$ and also satisfies the Laplace equation

$$\Delta u(x) = 0, \qquad \forall x \in \Omega$$

When Ω is open we speak of *interior regularity*. If Ω has C^∞ boundary and $u_0 \in C^\infty(\Omega)$, then $\overline{u}$ lives in fact in $C^\infty(\overline{\Omega})$ whereby we have *regularity up to the boundary*. In our study we will only deal with the former case.

Theorem 4.5 (Weyl's Lemma)**.** *Suppose we have an open set $\Omega \subset \mathbb{R}^n$ and a function $u \in L^1_{loc}(\Omega)$. If u satisfies*

$$\int_\Omega u(x)\Delta\psi(x)\,dx = 0, \qquad \forall\psi \in C_0^\infty(\Omega) \tag{4.5}$$

then $u \in C^\infty(\Omega)$ and $\Delta u = 0$ in Ω.

Remark. *Note that if a function, v, solves the weak Laplace equation, that is*

$$\int_\Omega \langle \nabla v(x)\,;\nabla \varphi(x)\rangle \; dx = 0, \qquad \forall \varphi \in W_0^{1,2}(\Omega) \tag{E$_w$}$$

then it automatically, solves (4.5)*. If* $v \in W^{1,2}(\Omega)$*, the converse also holds.*

In order to prove Weyl's Lemma we need another lemma first, whose proof is left as an exercise. For starter's we denote the ball of centre $x \in \Omega \subset \mathbb{R}^n$ and radius $R > 0$ with $B_R(x) = \{y \in \mathbb{R}^n \; : \; |y - x| < R\}$ and the area of the unit ball in $\mathbb{R}^n$ with $\alpha_{n-1} \equiv m_{n-1}(\partial B_1(0))$, where m_k is the k-Lebesgue measure in $\mathbb{R}^n$. Then, we have:

Lemma 4.6. *Let* $\Omega \subset \mathbb{R}^n$ *be open and* $u \in C(\Omega)$*. If u satisfies the so-called* mean value formula

$$u(x) = \frac{1}{\alpha_{n-} R^{n-1}} \int_{\partial B_R(x)} u \, dm_{n-1}$$

for every $x \in \Omega$ *and for every* $R > 0$ *so that it is* $B_R(x) \subset \Omega$*, then u is* $C^\infty(\Omega)$*.*

Proof of the theorem. Our purpose will be to show that the function

$$\overline{u}(x, R) \equiv \frac{1}{\alpha_{n-1} R^{n-1}} \int_{\partial B_R(x)} u \, dm_{n-1} \tag{4.6}$$

is independent of R (so $\overline{u}(x, R) = \overline{u}(x)$) and continuous. If these hold, we can integrate (4.6) with respect to R in the following way

$$(4.6) \implies n\alpha_{n-1} R^{n-1} \overline{u}(x) = n \int_{\partial B_R(x)} u \, dm_{n-1} \implies \alpha_{n-1} R^n \overline{u}(x) = \int_{B_R(x)} u dm_n$$

And so, we end up with

$$\overline{u}(x) = \frac{1}{m_n(B_R(x))} \int_{B_R(x)} u \, dm_n \tag{4.7}$$

which through Lebesgue's Differentiation Theorem, while R tends to 0, yields that $\overline{u} = u$ almost everywhere in Ω. Lemma 4.6 would tell us, though, that $\overline{u}$ is in fact $C^\infty(\Omega)$ allowing in turn an integration by parts on (4.5). In that case, the Fundamental Lemma of the Calculus of Variations 1.13 would immediately give that $\Delta u = 0$.

We now move towards the proof of our statements. In order to make progress, before everything else we need to find a convenient function $\psi \in C^\infty(\Omega)$. To begin with, chose a point $r \in (0, R)$ and a function $\varphi \in C^\infty(\Omega)$ with $\operatorname{supp} \varphi \subset (r, R)$. Then, the function

$$\psi(y) = \varphi(|y - x|)$$

lives in $C_0^\infty(B_R(x))$ and setting $\rho \equiv |y - x|$ we take

$$\Delta\psi(\rho) = \varphi''(\rho) + \frac{n-1}{\rho}\varphi'(\rho) = \frac{1}{\rho^{n-1}} \frac{d}{d\rho}\big(\rho^{n-1}\varphi'(\rho)\big)$$

The function $\Phi(\rho) \equiv \frac{d}{d\rho}\big(\rho^{n-1}\varphi'(\rho)\big)$ lives in $C_0^\infty(r, R)$ and satisfies $\int_r^R \Phi(\rho)\, d\rho = 0$. It is important to notice that the above procedure can be reversed. This means that given Φ we can always find a function $\varphi \in C^\infty(\mathbb{R})$ with $\operatorname{supp}\varphi \subset (r,R)$ such that $\Phi(\rho) = \frac{d}{d\rho}\big(\rho^{n-1}\varphi'(\rho)\big)$ which simultaneous implies that $\int_r^R \Phi(\rho)\,d\rho = 0$.

Taking $0 < r < R$ and an arbitrary $\Phi \in C_0^\infty(r,R)$ we can have $\varphi \in C^\infty$ as described above. Then, setting $\psi(y) = \varphi(|y-x|)$ it is $\psi = 0$ on $\Omega\backslash B_R(x)$ and so

$$0 = \int_\Omega u\Delta\psi\,dy = \int_{B_R(x)} u\Delta\psi\,dy \underset{\rho\equiv|y-x|}{=} \int_r^R \int_{\partial B_\rho(x)} \frac{u(\vartheta)}{\rho^{n-1}}\frac{d}{d\rho}\big(\rho^{n-1}\varphi'(\rho)\big)\,d\vartheta_{n-1}d\rho$$

$$= \alpha_{n-1}\int_r^R \Phi(\rho)\frac{1}{\alpha_{n-1}\rho^{n-1}}\int_{\partial B^\rho(x)} u(\vartheta)\,d\vartheta_{n-1}d\rho = \alpha_{n-1}\int_r^R \Phi(\rho)\overline{u}(x,\rho)\,d\vartheta_{n-1}d\rho$$

$$\Longrightarrow \int_r^R \Phi(\rho)\overline{u}(x,\rho)\,d\vartheta_{n-1}d\rho = 0, \qquad \text{with } \int_r^R \Phi(\rho)\,d\rho = 0$$

Corollary 1.14 tells us now that $\overline{u}(x,\rho)$ is constant for almost every $\rho \in (r,R)$ and so, $\overline{u}(x,\rho) = \overline{u}(x)$.

The only thing left to do is to prove the continuity of

$$\overline{u}(x) = \frac{1}{\alpha_{n-1}\rho^{n-1}}\int_{\partial B_\rho(x)} u\,dm_{n-1}$$

which comes directly from the formula (4.7). Indeed, if $\omega_n \equiv B_1(0)$ is the volume of the n-dimensional unit ball, then

$$|\overline{u}(x) - \overline{u}(y)| = \frac{1}{\omega_n R^n}\left|\int_{B_R(x)} u\,dm_n - \int_{B_R(y)} u\,dm_n\right| \le \frac{1}{\omega_n R^n}\int_{B_R(x)\triangle B_R(y)} |u|\,dm_n$$

(Recall that $A \triangle B = A\backslash B \cup B\backslash A$.) However, we already know by hypothesis that $u \in L^1_{loc}(\Omega)$ and so $u \in L^1(B_R(x) \triangle B_R(y))$. Theorem 1.17 yields, then, the desired continuity and the proof is complete. □

Theorem 4.7. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^{k+2} boundary and $f \in W^{k,2}(\Omega)$, where $k \ge 0$ is an integer. The problem*

$$\inf\left\{ I(u) = \int_\Omega \Big(\frac{1}{2}|\nabla u(x)|^2 - f(x)u(x)\Big)\,dx \ :\ u \in W_0^{1,2}(\Omega)\right\} \tag{P'}$$

admits a unique minimizer $\overline{u} \in W^{k+2,2}(\Omega)$. Additionally, there exists a constant $c > 0$, dependent on Ω and k, such that

$$\|\overline{u}\|_{W^{k+2,2}} \le c\,\|f\|_{W^{k,2}}$$

Remark. *i) Through the proof one can see that with the addition of an iteration argument we can see that $\overline{u}$ is in fact in $C^\infty(\overline{\Omega})$ if we set $k = +\infty$.*

ii) Further results can be proved. In particular, with extra boundary hypotheses and for $0 < \alpha < 1$ there are the Schauder estimates

$$\|\overline{u}\|_{C^{k+2,\alpha}} \le c\,\|f\|_{C^{k,\alpha}}$$

and the Calderon-Zygmund estimates for $1 < p < +\infty$

$$\|\overline{u}\|_{W^{k+2,p}} \le c\,\|f\|_{W^{k,p}}$$

In order to go through the proof of this theorem we will need another tool, which was introduced by Nirenberg. Let $h \in \mathbb{R}^n \backslash \{0\}$ and $u : \mathbb{R}^n \to \mathbb{R}$. For each x in $\mathbb{R}^n$ we define the *difference quotient* of u to be

$$(D_h u)(x) = \frac{u(x+h) - u(x)}{|h|}$$

It is straightforward to establish the following properties provided u is such that each symbol is well defined:

$$\nabla(D_h u) = D_h(\nabla u)$$
$$\|D_{-h} u\|_{L^2(\mathbb{R}^n)} \le \|\nabla u\|_{L^2(\mathbb{R}^n)}$$

Furthermore, appealing to Theorem 1.21 we have that whenever we can find a constant $\gamma > 0$ such that $\|D_h u\|_{L^2(\mathbb{R}^n)} \le \gamma$ the function u is in $W^{1,2}(\mathbb{R}^n)$.

Now, we can write down our proof.

Proof. We already know from Corollary 3.4 that there exists a minimizer of (P'), say $\overline{u} \in W_0^{1,2}(\Omega)$, which additionally satisfies

$$\int_\Omega \langle \nabla \overline{u}(x) \, ; \nabla \varphi(x) \rangle \; dx = \int_\Omega f(x) \varphi(x) \, dx, \qquad \forall \varphi \in W_0^{1,2}(\Omega) \tag{4.8}$$

As we said earlier, we will only deal with the interior regularity of $\overline{u}$. In particular, we will show that $\overline{u} \in W_{loc}^{k+2,2}(\Omega)$, which is equivalent to $\varphi \overline{u}$ living in $W^{k+2,2}(\Omega)$ for every $\varphi \in C_0^\infty(\Omega)$. If we, now, denote $v \equiv \varphi \overline{u}$, then v is in $W^{1,2}(\mathbb{R}^n)$. From (4.8), we can write that $\Delta \overline{u} = -f$ and thus

$$\Delta v = \Delta(\varphi \overline{u}) = \varphi \Delta \overline{u} + \overline{u} \Delta \varphi + 2 \langle \nabla \varphi \, ; \nabla \overline{u} \rangle = -\varphi f + \overline{u} \Delta \varphi + 2 \langle \nabla \varphi \, ; \nabla \overline{u} \rangle \equiv g$$

(All the above equalities are meant in the weak sense. The usual properties of derivation apply here as well.) We only have to show, then, that every $v \in W^{1,2}$ satisfying

$$\int_{\mathbb{R}^n} \langle \nabla v(x) \, ; \nabla \psi(x) \rangle \; dx = \int_{\mathbb{R}^n} g(x) \psi(x) \, dx, \qquad \forall \psi \in W^{1,2}(\mathbb{R}^n) \tag{4.9}$$

lives actually in $W^{k+2,2}(\mathbb{R}^n)$ whenever $g \in W^{k,2}(\mathbb{R}^n)$.

For starters, we will establish this statement for $k = 0$ and after that we can begin our familiar iteration process. Here is where the difference quotients appear on the scene. Consider the function

$$\overline{\psi}(x) \equiv (D_{-h}(D_h v))(x) = \frac{2v(x) - v(x+h) - v(x-h)}{|h|^2}$$

and observe that it lives in $W^{1,2}(\mathbb{R}^n)$, since $v \in W^{1,2}(\mathbb{R}^n)$. Substituting $\overline{\psi}$ into (4.9) we take that

$$\int_{\mathbb{R}^n} g(x)(D_{-h}(D_h v))(x) \, dx = \int_{\mathbb{R}^n} \langle \nabla v(x) \, ; \nabla (D_{-h}(D_h v))(x) \rangle \; dx$$

$$= \frac{1}{|h|^2} \int_{\mathbb{R}^n} \langle \nabla v(x) \, ; 2\nabla v(x) - \nabla v(x+h) - \nabla v(x-h) \rangle \; dx$$

$$= \frac{2}{|h|^2} \int_{\mathbb{R}^n} \langle \nabla v(x) \, ; \nabla v(x) \rangle \; dx - \frac{1}{|h|^2} \int_{\mathbb{R}^n} \langle \nabla v(x) \, ; \nabla v(x+h) \rangle \; dx - \frac{1}{|h|^2} \int_{\mathbb{R}^n} \langle \nabla v(x) \, ; \nabla v(x-h) \rangle \; dx$$

$$= \frac{2}{|h|^2}\int_{\mathbb{R}^n} |\nabla v(x)|^2\,dx - \frac{2}{|h|^2}\int_{\mathbb{R}^n} \langle \nabla v(x)\,;\nabla v(x+h)\rangle\;dx$$

$$= \frac{1}{|h|^2}\int_{\mathbb{R}^n}\Big(|\nabla v(x)|^2 + |\nabla v(x+h)|^2 - 2\,\langle \nabla v(x)\,;\nabla v(x+h)\rangle\Big)\,dx$$

$$= \frac{1}{|h|^2}\int_{\mathbb{R}^n} |\nabla v(x) - \nabla v(x+h)|^2\,dx = \int_{\mathbb{R}^n} |(D_h \nabla v)(x)|^2\,dx$$

$$\Longrightarrow \int_{\mathbb{R}^n} g(x)(D_{-h}(D_h v))(x)\,dx = \int_{\mathbb{R}^n} |(D_h\nabla v)(x)|^2\,dx$$

Hence, using the properties of D_h and with the help of Cauchy-Schwarz inequality, we get

$$\|D_h\nabla v\|_{L^2}^2 = \int_{\mathbb{R}^n} |(D_h\nabla v)(x)|^2\,dx = \int_{\mathbb{R}^n} g(x)(D_{-h}(D_h v)(x)\,dx$$

$$\overset{\text{C-S}}{\leq} \|g\|_{L^2}\,\|D_{-h}(D_h v)\|_{L^2} \leq \|g\|_{L^2}\,\|\nabla(D_h v)\|_{L^2} = \|g\|_{L^2}\,\|D_h\nabla v\|_{L^2}$$

$$\Longrightarrow \|D_h\nabla v\|_{L^2}^2 \leq \|g\|_{L^2}\,\|D_h\nabla v\|_{L^2}$$

$$\Longrightarrow \|D_h\nabla v\|_{L^2} \leq \|g\|_{L^2}$$

This last inequality tells us that $\nabla u \in W^{1,2}(\mathbb{R}^n;\mathbb{R}^n)$ which in turn implies that $u \in W^{2,2}(\mathbb{R}^n)$.

Now, we proceed to the case where $k = 1$. Notice that v being in $W^{2,2}(\mathbb{R}^n)$ allows us to use its second weak derivatives and in particular we can prove that the function ∇v_{x_i} also solves (4.9). To simplify our calculations we consider the test functions living only in $C_0^\infty(\mathbb{R}^n)$, since they are dense in $W^{1,2}(\mathbb{R}^n)$ anyway. The function g lives in $W^{1,2}(\mathbb{R}^n)$ and so its weak derivatives, g_{x_i}, live in L^2. We, then, have

$$\int_{\mathbb{R}^n} \langle \nabla v_{x_i}\,;\nabla\psi\rangle\;dx = \int_{\mathbb{R}^n} \langle (\nabla v)_{x_i}\,;\nabla\psi\rangle\;dx = -\int_{\mathbb{R}^n} \langle \nabla v\,;\nabla\psi_{x_i}\rangle\;dx = -\int_{\mathbb{R}^n} g\psi_{x_i}\,dx = \int_{\mathbb{R}^n} g_{x_i}\psi\,dx$$

This means that $\nabla v_{x_i} \in W^{1,2}(\mathbb{R}^n)$ for all $i = 1, 2, \ldots, n$ and thus $v \in W^{3,2}(\mathbb{R}^n)$. Repeating this argument for the higher derivatives that occur completes the proof. □

4.4 General Results

In this section we are going to present the main theorems that generalize the Dirichlet case. Their proof, however, is a rather difficult task which refrains from the purposes of this study.

Theorem 4.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $f \in C^\infty(\Omega\times\mathbb{R}\times\mathbb{R}^n)$. We will denote the vector $(f_{x_1}, f_{x_2}, \ldots, f_{x_n})$ with f_x and the vector f_ξ with $(f_{\xi_1}, f_{\xi_2}, \ldots, f_{\xi_n})$. Similarly for higher derivatives. Suppose that the following hold:*

$$a_1(1+|u|^2+|\xi|^2)^{\frac{p}{2}} - a_2 \leq f\,(x,u,\xi) \leq a_3(1+|u|^2+|\xi|^2)^{\frac{p}{2}}$$

$$|f_\xi|, |f_{x\xi}|, |f_u|, |f_{xu}| \leq a_3(1+|u|^2+|\xi|^2)^{\frac{p-1}{2}}$$

$$|f_{u\xi}|, |f_{uu}| \leq a_3(1+|u|^2+|\xi|^2)^{\frac{p-2}{2}}$$

$$\alpha|\lambda|^2(1+|u|^2+|\xi|^2)^{\frac{p-2}{2}} \leq \sum_{i,j=1}^{n} \lambda_i\lambda_j f_{\xi_i\xi_j}(x,u,\xi) \leq \beta|\lambda|^2(1+|u|^2+|\xi|^2)^{\frac{p-2}{2}}$$

for some positive constants $a_1, a_2, a_3, \alpha, \beta > 0$*, where* $p \geq 2$*, and some* $\lambda \in \mathbb{R}^n$*. Then, any minimizer of the problem*

$$\inf\left\{I(u) = \int_\Omega f(x, u(x), \nabla u(x))\, dx \;:\; u \in u_0 + W_0^{1,p}(\Omega)\right\} = m \tag{P}$$

lives in $C^\infty(F)$ *for every* $F \subset \overline{F} \subset \Omega$*.*

Theorem 4.9. *Suppose* $\Omega \subset \mathbb{R}^n$ *is a bounded open set and let* $\overline{v} \in W^{1,2}(\Omega)$ *be a solution of the equation*

$$\sum_{i,j=1}^{n} \int_\Omega a_{ij}(x) v_{x_i}(x) \varphi_{x_j}(x)\, dx = 0, \qquad \forall \varphi \in W_0^{1,2}(\Omega)$$

where a_{ij} *are functions in* $L^\infty(\Omega)$*. If* $c > 0$ *is a constant such that*

$$\sum_{i,j=1}^{n} \lambda_i\lambda_j a_{ij}(x) \geq c|\lambda|^2, \qquad \forall \lambda \in \mathbb{R}^n \quad \text{almost everywhere in } \Omega$$

then there exists $1 > \alpha > 0$ *so that* $\overline{v} \in C^{0,\alpha}(F)$ *for every* $F \subset \overline{F} \subset \Omega$*.*

It is advisable to take some time comparing the last two theorems. For instance, observe that in the case where f doesn't depend explicitly on x and u, that is $f(x, u, \xi) = f(\xi)$, the coefficients $a_{ij}(x)$ of Theorem 4.9 are the equivalent of $f_{\xi_i\xi_j}(\nabla u(x))$ of Theorem 4.8, while $\overline{v} = \overline{u}_{x_i} \in W^{1,2}(\Omega)$, $\overline{u}$ being a minimizer of (P).

Finally, we have the unfortunate result that if $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ with $n, m > 1$, only partial regularity can be generally obtained. But this is a story for another time.