Structural results for multiple translational tilings

Nikos Poursalidis

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Contents

Prologue

A tiling of $\mathbb{R}^d$ is a collection of sets, called tiles, which have disjoint interiors and whose union is the entire $\mathbb{R}^d$. Here we are interested in tilings, where all tiles are translates of a single tile. Also the only allowed motions of the tile are translations, not rotations or reflections. The word multiple, in the title of this thesis, refers to the fact that we allow a tiling to have many levels, i.e. we can cover each point several times, but we insist that all points are covered the same number of times. Its easy to see that the only polygons that tile the plane under translations at level 1, are parallelograms and hexagons. However, octagons can, also, tile the plane, but at a higher level, and so can many other shapes (See Figure 2). The purpose of this thesis is to present some results on the structure of multiple translational tilings. This means that we are trying to describe how the underlying set of translations of a tiling, under some assumptions for the tile, looks like. We, mostly, use tools from Harmonic Analysis. Almost all results that we present here, can be found in the bibliography [5], [9], [10].

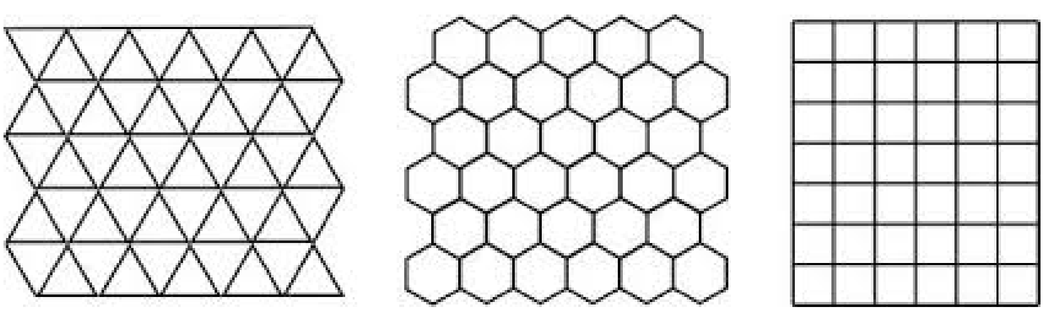

Figure 1: Here we have tilings by regular polygons. But the triangle doesn't tile the plane under translations, unlike squares and hexagons.

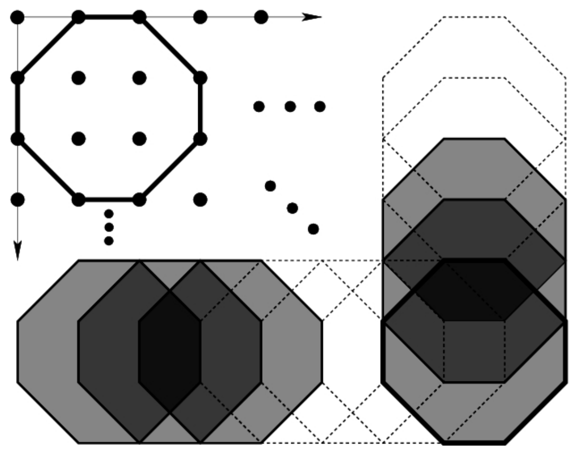

Figure 2: An octagon that 7-tiles, but does not 1-tile. Here, each point, not on the boundary of one of the tiles, is covered exactly 7 times, once we translate the octagon by all of the integer translation vectors.

Chapter 1

Introduction

1.1 Basic concepts

We now give the formal definition of a multiple translational tiling in $\mathbb{R}^d$.

Definition 1.1.1 (Translational tiling). *Suppose $f \in L^1(\mathbb{R}^d)$ is a non-negative function and $\Lambda \subseteq \mathbb{R}^d$ is a discrete multiset. We say that f tiles $\mathbb{R}^d$ with tile set Λ and weight $w \in \mathbb{R}$ if*

$$\sum_{\lambda\in\Lambda} f(x-\lambda) = w, \tag{1.1}$$

almost everywhere and we write $f + \Lambda = w\mathbb{R}^d$.

In the particular case when $f = \mathbf{1}_\Omega$ is the indicator function of a measurable set $\Omega \subseteq \mathbb{R}^d$, we have the classical geometric situation and we write $\Omega + \Lambda = l\mathbb{R}^d$, where l is an integer that denotes the level of the tiling.
The assumption tiling $f + \Lambda = w\mathbb{R}^d$, has some immediate implications about the density properties of the multiset Λ.

Definition 1.1.2 (Uniform density). *A multiset $\Lambda \subseteq \mathbb{R}^d$ has asymptotic density ρ if*

$$\lim_{R\to\infty} \frac{|\Lambda \cap B_R(x)|}{|B_R(x)|} = \rho$$

*uniformly in $x \in \mathbb{R}^d$, where $B_R(x)$ is the ball of radius R centred at x.
We write $\mathrm{dens}\Lambda = \rho$. We say that Λ has (uniformly) bounded density if the fraction above is bounded by a constant ρ uniformly for $x \in \mathbb{R}^d$ and $R > 1$. We say then that Λ has density (uniformly) bounded by ρ.*

Lemma 1.1.1. *If $f \in L^1(\mathbb{R}^d)$ is a non negative function with $\int f > 0$ and $\Lambda \subseteq \mathbb{R}^d$ is a multiset such that $f + \Lambda = w\mathbb{R}^d$, then Λ has bounded density.*

Proof. By hypothesis
$$\sum_{\lambda \in \Lambda} f(x-\lambda) = w$$
almost everywhere and $w > 0$ (since $\int f > 0$). We choose $R > 1$ such that $I = \int_{B_R(0)} f > 0$. Let $x_0 \in \mathbb{R}^d$ be arbitrary, we have

$$\begin{aligned} |B_{2R}(0)|w &= \int_{B_{2R}(x_0)} \sum_{\lambda \in \Lambda} f(x-\lambda)dx \\ &\geq \int_{B_{2R}(x_0)} \sum_{|\lambda - x_0| < R} f(x-\lambda)dx \\ &\geq |\Lambda \cap B_R(x_0)| \int_{B_R(0)} f(x)dx. \end{aligned}$$

Thus $|\Lambda \cap B_R(x_0)| \leq |B_{2R}(0)|w/I$ is bounded independent of $x_0 \in \mathbb{R}^d$, which implies that Λ has uniformly bounded density. $\square$

1.2 Basic tools and connection with tilings

In this section we present, briefly, the notions and tools we will need. The main reference is Rudin [1].

The Fourier transform of a function $f \in L^1(\mathbb{R}^d)$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle \xi, x \rangle} dx, \quad \xi \in \mathbb{R}^d.$$

We also have the Fourier inversion Formula which says that, if $\hat{f} \in L^1(\mathbb{R}^d)$, then
$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i \langle \xi, x \rangle} d\xi \quad a.e.$$

Definition 1.2.1 (Schwartz space)**.** *We define the Schwartz space as*

$$\mathcal{S}(\mathbb{R}^d) = \{\phi \in C^\infty(\mathbb{R}^d) : \sup_{x \in \mathbb{R}^d} |x^\beta D^\alpha \phi(x)| < +\infty, \quad \forall \alpha, \beta \ \in \mathbb{N}_0^d\},$$

where $x^\beta = x^{\beta_1} ... x^{\beta_d}$ *and* $D^\alpha = (\frac{\partial}{\partial x_1})^{\alpha_1} ... (\frac{\partial}{\partial x_d})^{\alpha_d}$.

We also define the sequence of seminorms in $C^\infty(\mathbb{R}^d)$ by

$$p_k(\phi) = \sup_{|\alpha|\leq k} \sup_{x\in\mathbb{R}^d} (1+|x|^2)^{k/2}|D^\alpha\phi(x)|,$$

where $|\alpha| = \alpha_1 + ... + \alpha_d$.

From these definitions it follows that $\phi \in \mathcal{S}(\mathbb{R}^d)$ if and only if $p_k(\phi) < +\infty, \forall \alpha \in \mathbb{N}_0^d$. Moreover these semi-norms induce a metrizable topology in Schwartz space, which under the translation invariant metric

$$d(\phi,\psi) = \sum_{k=0}^{+\infty} \frac{1}{2^k} \frac{p_k(\phi-\psi)}{1+p_k(\phi-\psi)},$$

becomes a Frechet space, i.e. a locally convex, complete metric space.

Observation 1.2.1. *It is obvious that*

$$C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{S}(\mathbb{R}^d) \subseteq L^1(\mathbb{R}^d),$$

thus the Schwartz space is a dense subspace of $L^1(\mathbb{R}^d)$ *and it is also known that the Fourier transform is an automorphism in Schwartz space.*

Now the space of tempered distributions, denoted by $\mathcal{S}'(\mathbb{R}^d)$, is the dual space of Schwartz space, equipped with the *weak**-topology induced by the topology of $\mathcal{S}(\mathbb{R}^d)$. For convenience we use the symbols $\mathcal{S}$, $\mathcal{S}'$ instead of $\mathcal{S}(\mathbb{R}^d)$ $\mathcal{S}'(\mathbb{R}^d)$, respectively.

We now define some operations in the space of tempered distributions by duality.

(i) Fourier Transform:

$$\hat{u}(\phi) = u(\hat{\phi})$$

where $u \in \mathcal{S}'$, $\phi \in \mathcal{S}$.

(ii) Convolution with function:

$$(\phi * u)(x) = u(\tau_x\tilde{\phi})$$

where $u \in \mathcal{S}'$, $\phi \in \mathcal{S}$ and $\tilde{\phi} = \bar{\phi}(-x)$, $\phi * u$ is a function on $\mathcal{S}$.

(iii) Differentiation:

$$(D^\alpha u)(\phi) = (-1)^{|\alpha|}u(D^\alpha\phi)$$

where $u \in \mathcal{S}'$, $\phi \in \mathcal{S}$, $\alpha \in \mathbb{N}_0^d$.

(iv) Multiplication by function:

$$(fu)(\phi) = u(\bar{f}\phi),$$

where $u \in \mathcal{S}'$, $\phi \in \mathcal{S}$, $f \in C^\infty(\mathbb{R}^d)$.

The support of $u \in \mathcal{S}'$, denoted by supp$u$, is defined as the smallest closed set $X \subseteq \mathbb{R}^d$ such that:

$$u(\phi) = 0, \quad \forall \phi \in C_0^\infty(\mathbb{R}^d \setminus X).$$

We identify locally integrable functions and Radon measures with distributions (the dual space of the space of test functions $C_c^\infty(\mathbb{R}^d)$) with:

(i) $f \longleftrightarrow u_f$:

$$u_f(\phi) = \int \bar{\phi}(x) f(x) dx,$$

(ii) $\mu \longleftrightarrow u_\mu$:

$$u_\mu(\phi) = \int \bar{\phi}(x) d\mu(x).$$

We are particularly interested in Radon measures consisting of point masses at a discrete set Λ, i.e.

$$\delta_\Lambda = \sum_{\lambda \in \Lambda} \delta_\lambda$$

Remark 1.2.1. *The Radon measure δ_Λ is not necessarily a tempered distribution. A sufficient condition for δ_Λ to be a tempered distribution is the growth condition*

$$|\Lambda \cap B_R(0)| \ll R^c, \quad as \quad R \to \infty, \tag{1.2}$$

for some $c > 0$. Where the symbol $f(x) \ll g(x)$ denotes that there is a positive constant C such that $f(x) \leq Cg(x)$ for the indicated range of x.
A set of bounded density satisfies (1.2), with $c = d$, the dimension.

Definition 1.2.2 (Lattice)**.** *A lattice is a discrete subgroup of a topological abelian group.*

When Λ is a lattice of full rank in $\mathbb{R}^d$, then $\Lambda = A\mathbb{Z}^d$, where $A \in GL(d, \mathbb{R})$. In this case, the Fourier transform of the tempered distribution δ_Λ has a simple and useful form given by Poisson Summation Formula(PSF)

$$\hat{\delta_\Lambda} = \frac{1}{\mathrm{detA}}\delta_{\Lambda^*},$$

where

$$\Lambda^* = \{x \in \mathbb{R}^d : \langle x, \lambda\rangle \in \ \mathbb{Z}, \quad \forall\lambda \in \Lambda\} = A^{-\top}\mathbb{Z}^d$$

is the dual lattice of Λ.

Now we argue informally in order to show the connection between harmonic analysis and tilings. Suppose that f tiles $\mathbb{R}^d$ with tile set Λ and weight w, so from (1.1) we have

$$\sum_{\lambda\in\Lambda} f(x-\lambda) = w, \quad a.e.,$$

so that now the tiling condition can be rewritten as convolution

$$f * \delta_\Lambda = w,$$

and, taking the Fourier Transform, this gives

$$\hat{f}\hat{\delta_\Lambda} = w\delta_0.$$

From this we conclude that

$$\mathrm{supp}\hat{\delta}_\Lambda \subseteq \{0\} \cup \mathcal{Z}(\hat{f}) \tag{1.3}$$

where $\mathcal{Z}(\hat{f}) = \{x \in \mathbb{R}^d : \hat{f}(x) = 0\}$ is the zero set of $\hat{f}$.

Remark 1.2.2. *The relation (1.3) is the connection between harmonic analysis and translational tilings. However, neither $f * \delta_\Lambda$ nor $\hat{f}\hat{\delta_\Lambda}$ is well defined, when $f \in L^1(\mathbb{R}^d)$ in general. We are going to deal this problem in the next chapter.*

1.3 The dual group and Meyer's theorem

We start this section by giving some definitions, we follow Rudin [2]. Suppose G is a locally compact abelian (LCA) group. A complex function $\gamma : G \to \mathbb{C}$ is called a character of G if

(i) $|\gamma(x)| = 1, \quad \forall x \in G$

(ii) $\gamma(x+y) = \gamma(x)\gamma(y), \quad \forall x, y \in G$

The dual group of a LCA group G is the set of all continuous characters of G. We denote the action of a character γ to an an element $x \in G$ as $\langle x, \gamma \rangle$, i.e. $\langle x, \gamma \rangle = \gamma(x)$.

The Fourier transform of a function $f \in L^1(G)$ is continuous function $\hat{f}$ defined on Γ by

$$\hat{f}(\gamma) = \int_G f(x) \langle -x, \gamma \rangle dx.$$

Notice $\langle -x, \gamma \rangle = \bar{\gamma}(x)$.
If μ is a finite Borel measure on G its Fourier transform is the continuous function $\hat{\mu}$ defined on Γ by

$$\hat{\mu}(\gamma) = \int_G \langle -x, \gamma \rangle d\mu(x)$$

where the integration carried out with respect to the Haar measure on G. The normalization of the Haar measure will usually be implicit.

Suppose Γ is the dual group of the LCA group G. We denote Γ_d the group Γ equipped with the discrete topology and $\bar{G}$ the dual group of Γ_d. Then $\bar{G}$ is a compact abelian group which we call the Bohr compactification of G. Let $\tau : G \to \bar{G}$ be defined as $x \longrightarrow \tau_x$, where $\langle \gamma, \tau_x \rangle = \langle x, \gamma \rangle$.
The map τ is a continuous isomorphism from G onto a dense subgroup $\tau(G)$ of $\bar{G}$. Hence this map allows us to regard G as a dense subgroup $\bar{G}$, so that $\bar{G}$ is indeed a compactification of G.

Definition 1.3.1 (Coset ring)**.** *The* coset ring *of a LCA group G is the smallest algebra(family of sets which is closed under finite set-theoretic operations) generated by all open cosets of G.*

Theorem 1.3.1 (Cohen's idempotent theorem)**.** *[3] Let G be a LCA group and Γ its dual group. If μ is a finite Borel measure on G and $\hat{\mu}$ takes only finite many values, i.e. the range of $\hat{\mu}$ is a finite set $S = \{s_1, ..., s_n\} \subseteq \mathbb{C}$, then the preimage of s_j*

$$\hat{\mu}^{-1}(s_j) = \{\gamma \in \Gamma : \hat{\mu}(\gamma) = s_j\}$$

is in the coset ring of Γ, for each $j = 1, ..., n$

The next theorem, whose proof is based on Cohen's idempotent theorem, is the cornerstone of this thesis, since it provides information on the structure of the set of translations $\Lambda \subseteq \mathbb{R}^d$.

Theorem 1.3.2 (Meyer's theorem). *[5] Let $\Lambda \subseteq \mathbb{R}^d$ be a discrete set and δ_Λ be the Radon measure*

$$\delta_\Lambda = \sum_{\lambda \in \Lambda} c_\lambda \delta_\lambda, \quad c_\lambda \in S,$$

where $S \subseteq \mathbb{C} \setminus \{0\}$ is a finite set.
Suppose that δ_Λ is a tempered distribution and that $\hat{\delta_\Lambda}$ is a Radon measure on $\mathbb{R}^d$ which satisfies the growth condition

$$|\hat{\delta_\Lambda}|(B_R(0)) \ll R^d, \quad as \quad R \longrightarrow \infty.$$

Then, for each $s \in S$, the set

$$\Lambda_s = \{\lambda \in \Lambda : c_\lambda = s\}$$

is in the coset ring of $\mathbb{R}^d$.

Proof. Let $\phi \in C_c^\infty(B_1(0))$ such that $\phi(0) = 1$ and $|\hat{\phi}(\xi)| \ll |\xi|^{-\alpha}$ for all $\alpha > 0$. We define the sequence of functions

$$\mu_n(x) = \phi(nx) * \delta_\Lambda(x)$$

and taking the Fourier transform we have

$$\hat{\mu_n}(\xi) = \frac{1}{n^d}\hat{\phi}(\xi/n)\hat{\delta_\Lambda}(\xi).$$

We claim that $\hat{\mu_n}$ is a uniformly bounded sequence of measures in $\mathbb{R}^d$.
Indeed,

$$|\hat{\mu_n}|(B_n(0)) \leq \frac{1}{n^d}\left\|\hat{\phi}\right\|_\infty |\hat{\delta_\Lambda}(B_n(0))| \ll 1, \quad as \quad R \longrightarrow \infty$$

since $|\hat{\delta_\Lambda}|(B_R(0)) \ll R^d \quad as \quad R \longrightarrow \infty$.
Furthermore, if $2^k \gg n$ and using the fact that $|\hat{\phi}(\xi)| \ll |\xi|^{-d-1}$, as $\xi \longrightarrow \infty$, we have

$$|\hat{\mu_n}|(B_{2k+1}(0) \setminus (B_{2k}(0)) \ll \frac{1}{n^d}|\hat{\phi}|_{B_{\frac{2k+1}{n}}(0)\setminus B_{\frac{2k}{n}}(0)}|\hat{\delta_\Lambda}|(B_{2^k+1}(0))$$

$$\ll \frac{1}{n^d}(\frac{2^k}{n})^{-d-1}2^{(k+1)d} \ll n2^{-k}$$

Hence,

$$|\hat{\mu_n}|(B_n(0)^c) \ll \sum_{2^k \geqslant n} |\hat{\mu_n}|(B_{2^{k+1}}(0) \setminus (B_{2^k}(0)) \ll n \sum_{2^k \geqslant n} 2^{-k} \ll 1$$

Therefore $|\hat{\mu_n}|(\mathbb{R}^d) \ll 1$. We also notice that

$$lim_{n\to\infty}\mu_n(x) = c_x,$$

if $x \in \Lambda$ and is 0 otherwise. This is a consequence of the fact that Λ is discrete and the intersection $\cap_{n\in\mathbb{N}}\text{supp}\phi(nx) = \{0\}$.
We now use the Bohr compactification $\bar{\mathbb{R}^d}$ of $\mathbb{R}^d$. From definitions above we have that $\mathbb{R}^d$ is a dense subspace of the compact topological space $\bar{\mathbb{R}^d}$ and identifying the continuous functions on $\bar{\mathbb{R}^d}$ with the bounded continuous functions on $\mathbb{R}^d$, we get that

$$C(\bar{\mathbb{R}^d}) \subseteq C(\mathbb{R}^d) \cap L^\infty(R^d)$$

is a Banach space inclusion.
Since the measures $\hat{\mu_n}$ are uniformly bounded measures, they act on all bounded continuous functions of $\mathbb{R}^d$, and consequently on all continuous functions of $\bar{\mathbb{R}^d}$. That is they constitute a uniformly bounded family of linear functionals on $C(\bar{\mathbb{R}^d})$.By the Banach-Alaoglu theorem there exists a measure ν on $\bar{\mathbb{R}^d}$, such that for every function $f \in C(\bar{\mathbb{R}^d})$, there is a subsequence of $\hat{\mu_n}$ (call it again $\hat{\mu_n}$) such that

$$\langle f, \hat{\mu_n}\rangle \longrightarrow \langle f, \nu\rangle, \quad as \quad n \to \infty$$

Applying this with each character of $\bar{\mathbb{R}^d}$ in place of f we obtain

$$\hat{\nu}(x) = \lim_{n\to\infty} \hat{\hat{\mu_n}}(x) = 2\pi c_{-x}$$

if $-x \in \Lambda$ and is 0 otherwise. Hence $\hat{\nu}$ has finite range $2\pi S$. By Cohen's idempotent theorem the set $-\Lambda$, and thus Λ, belongs to the coset ring of $\mathbb{R}^d_d$, the d-dimensional Euclidean space with the discrete topology. □

Meyer's theorem would be useless if we are not able to describe the coset ring of $\mathbb{R}^d$. But we have the following results.

Theorem 1.3.3 (Rosenthal)**.** *[4] The elements of the coset ring of $\mathbb{R}_d$ which are discrete in the usual topology of $\mathbb{R}$ are precisely the sets of the form*

$$F \triangle \bigcup_{j=1}^{k} (\alpha_j \mathbb{Z} + \beta_j).$$

Where $F \subseteq \mathbb{R}$ is finite, $\alpha_j > 0$ and $\beta_j \in \mathbb{R}$, and $\triangle$ denotes symmetric difference.

Theorem 1.3.4 (Kolountzakis)**.** *[9] Every discrete element S of the coset ring of $\mathbb{R}^2$ may be written in the form*

$$S = \bigcup_{j=1}^{J} (A_j \setminus (B_1^j \cup ... \cup B_n^j)) \cup \bigcup_{l=1}^{L} L_l \triangle F$$

where A_j are two-dimensional translated lattices with pairwise intersections have dimension at most one, B_i^j, L_l are one-dimensional translated lattices and $F \subseteq \mathbb{R}^2$ is a finite set.

Chapter 2

Structure in dimension 1

2.1 Preliminaries

In this Chapter we give a result on the structure of tilings of the real line by a function, under some assumptions. In particular, we show that a function $f \in L^1(\mathbb{R})$ with compact support satisfies these assumptions. The main reference is M.N. Kolountzakis and J.C. Lagarias [5]. We will need the following lemma.

Lemma 2.1.1. *If $f \in L^1(\mathbb{R})$ and $A \subseteq \mathbb{R}$ is any set of bounded density, then*

$$G(x) = \sum_{a \in A} f(x-a)$$

is absolutely convergent almost everywhere and locally integrable.

Proof. Since A has bounded density there is a constant $C > 0$ such that $|A \cap (T, T+1)| \leq C$ for all $T \in \mathbb{R}$. Then

$$\begin{aligned}
\int_T^{T+1} |\sum_{a\in A} f(x-a)|dx &\leq \int_T^{T+1} \sum_{a\in A} |f(x-a)|dx \\
&\leq \sum_{a\in A} \int_T^{T+1} (|f(x-\lfloor a\rfloor) + f(x-\lfloor a\rfloor-1)|)dx \\
&\leq 2C \sum_{n\in\mathbb{Z}} \int_T^{T+1} |f(x-n)|dx \\
&= 2C \int_{\cup_{n\in\mathbb{Z}}[T-n,T-n+1]} |f(y)|dy \\
&= 2C \int_{-\infty}^{+\infty} |f(x)|dx = 2C \|f\|_1 < \infty
\end{aligned}$$

This proves local integrability and implies that $G(x)$ is defined as an absolutely convergent series almost everywhere. □

□

2.2 A spectral condition for tiling

Now we give a rigorous statement in $\mathbb{R}$ of the informal argument in the previous chapter that led us to the relation (1.3) which is the connection between harmonic analysis and tilings.

Theorem 2.2.1. *[5] Let $f \in L^1(\mathbb{R})$ have a Fourier Transform $\hat{f} \in C^\infty(\mathbb{R})$, and $A \subseteq \mathbb{R}$ be a set of bounded density.*

(i) If $f + A = w\mathbb{R}$, then

$$\mathrm{supp}\hat{\mu}_A \subseteq \{0\} \cup \mathcal{Z}(\hat{f})$$

(ii) If $\hat{\mu}_A$ is a Radon measure and if $\mathrm{supp}\hat{\mu}_A \subseteq \{0\} \cup \mathcal{Z}(\hat{f})$, then

$$f + A = w\mathbb{R}$$

for some weight w.

Proof. (i) We note that $B = \{0\} \cup \mathcal{Z}(\hat{f})$ is a closed set, as the inverse image of a singleton under the continuous function $\hat{f}$ union a singleton in a Hausdorff space. To prove that $\mathrm{supp}\hat{\mu}_A \subseteq B$ we need to show that

$$\hat{\mu}_A(\phi) = 0, \quad \forall \phi \in C_c^\infty(\mathbb{R} \setminus B)$$

(because $C_c^\infty(\mathbb{R} \setminus B)$ is dense in $\mathcal{S}(\mathbb{R} \setminus B)$ and $\hat{\mu}_A$ is continuous). We fix such a ϕ and take $\psi \in \mathcal{S}$ such that $\hat{\psi} \in C_c^\infty(\mathbb{R}), \hat{\psi}(0) = 1$ and $\hat{\psi}(\xi) \neq 0$ in an open set that contains $\mathrm{supp}\phi$.
We introduce the function

$$F = \psi * f$$

We note that since $\hat{\psi}$ has compact support and $\hat{f}$ is smooth, the product $\hat{\psi}\hat{f}$ is in Schwartz space, and so is then $F = \psi * f$, because Fourier transform is an automorphism in Schwartz space. Hence the convolution $F * \mu_A$ is well defined. Firstly we want to prove that

$$F * \mu_A = w$$

Indeed, $(F * \mu_A)(x) = \sum_{a \in A} F(x - a)$, with the sum converging absolutely since $F \in \mathcal{S}$ and

$$\begin{aligned}
\sum_{a \in A} F(x-a) &= \sum_{a \in A} \int_{-\infty}^{+\infty} f(x-a-t)\psi(t)dt \\
&= \int_{-\infty}^{+\infty} \psi(t) \sum_{a \in A} f(x-a-t)dt \\
&= w \int_{-\infty}^{+\infty} \psi(t)dt = w\hat{\psi}(0) = w
\end{aligned}$$

The interchange of summation and integration is justified using Fubini's theorem. For this we need $\sum_{a \in A} \int_{-\infty}^{+\infty} f(x-a-t)|\psi(t)|dt < \infty$, but this follows from the fact that $|\psi(t)|$ decreases faster than any power of t and the set A is of bounded density.

Taking the Fourier transform of $F * \mu_A = w$,we get

$$\hat{F}\hat{\mu}_A = w\delta_0$$

We note that $\hat{F} = \hat{\psi}\hat{f}$ and $\hat{f}$ have the same zero set within an open interval contain the support of ϕ. It follows that $\phi/\bar{\hat{F}}$ is a smooth function whose compact support is disjoint from B. We have

$$\hat{\mu}_A(\phi) = \hat{F}\hat{\mu}_A(\phi/\bar{\hat{F}}) = w\delta_0(\phi/\bar{\hat{F}}) = 0$$

since $0 \notin supp\phi$. This proves (i).

(ii) Let $\psi \in \mathcal{S}$ be arbitrary and define $F = \psi * f$. We want to show that supp$\hat{F}\hat{\mu}_A$ is $\{0\}$ for this we consider $\phi \in C_c^\infty((0, +\infty))$. We have

$$\hat{F}\hat{\mu}_A(\phi) = \hat{\mu}_A(\phi\bar{\hat{F}}) \tag{2.1}$$

But $\hat{F} = \hat{\psi}\hat{f}$ vanishes whenever $\hat{f}$ does. Since supp$\hat{\mu}_A \subseteq \{0\} \cup \mathcal{Z}(\hat{f})$ and supp$\hat{\mu}_A$ is a locally finite measure the second term in (2.1) is 0. So we conclude that supp$\hat{F}\hat{\mu}_A$ has no support in $(0, +\infty)$ and using a similar argument for an arbitrary $\phi \in C_c^\infty((-\infty, 0)$ we have that

$$\text{supp}(\hat{F}\hat{\mu}_A) = \{0\}$$

Therefore,

$$\hat{F}\hat{\mu}_A = (\hat{F}(0)\hat{\mu}_A(\{0\}))\delta_0 = (\hat{\psi}(0)\hat{F}(0)\hat{\mu}_A(\{0\}))\delta_0$$

and from the Fourier inversion theorem, since $\hat{1} = \delta_0$, we have

$$F * \mu_A = \hat{F}(0)\hat{\mu}_A(\{0\})\hat{\psi}(0) = w\int_{-\infty}^{+\infty} \psi(t)dt,$$

for $w = \hat{F}(0)\hat{\mu}_A(\{0\})$.
By Lemma 2.1.1 the function $G(x) = \sum_{a\in A} f(x-a)$ is a locally integrable function and we need to prove $G(x) = w$ almost everywhere. For all $x \in \mathbb{R}$ we have

$$\begin{aligned} w\int_{-\infty}^{+\infty} \psi(t)dt = F * \mu_A &= \sum_{a\in A} F(x-a) \\ &= \sum_{a\in A}\int_{-\infty}^{+\infty} f(x-a-t)\psi(t)dt \\ &= \int_{-\infty}^{+\infty} \psi(t)\sum_{a\in A} f(x-a-t)dt \end{aligned}$$

with the interchange of summation and integration justified as in part (i). Since $\psi \in \mathcal{S}$ is arbitrary, we have the equality

$$\sum_{a\in A} f(x-a-t) = w,$$

between functions (of t)as tempered distributions. Hence we have that the function $\sum_{a\in A} f(x-a-t)$ is ,also, equal to the constant function w almost for every $t \in \mathbb{R}$ and therefore $\sum_{a\in A} f(x-a) = w$, for almost all $x \in \mathbb{R}$.

□

2.3 Structure theorem for tile sets

We prove, under some conditions on $f \in L^1(\mathbb{R})$, that all tile sets of bounded density for f are finite unions of complete arithmetic progressions and subsequently we show that every compactly supported function f satisfies these conditions. This result extends (and proved earlier by Leptin and Müller in [7])that a bounded region admits only periodic tilings, which was proved by Lagarias and Wang in [6].

Theorem 2.3.1 (Structure theorem)**.** *[5] Let $f \in L^1(\mathbb{R})$ have a Fourier Transform $\hat{f} \in C^\infty(\mathbb{R})$ which has a discrete zero set satisfying the bound*

$$|\{\xi : \hat{f}(\xi) = 0, \quad |\xi| \le R\}| \le cR \tag{2.2}$$

for some constant c.
Suppose that f tiles $\mathbb{R}$ with the tile set A of bounded density, i.e. $f + A = w\mathbb{R}$ for some weight w.Then the set A is a finite union of complete arithmetic progression

$$A = \bigcup_{j=1}^{J} (\alpha_j \mathbb{Z} + \beta_j), \quad all \quad \alpha_j \ne 0.$$

Proof. Firstly we will show that the measure μ_A satisfies the hypotheses of Meyer's theorem and then from Rosenthal's theorem will obtain a structural result for the tile set A. By theorem 2.1(i) we have

$$\mathrm{supp}\hat{\mu}_A \subseteq B = \{0\} \cup \mathcal{Z}(\hat{f})$$

Hence the support of the tempered distribution $\hat{\mu}_A$ is a discrete set, since $\mathcal{Z}(\hat{f})$ is discrete by hypothesis.

Step 1: The tempered distribution $\hat{\mu}_A$ is a Radon measure.

From Rudin [1] (Theorem 6.25) we have that a distribution supported at a single point $b \in R$ is a finite linear combination of derivatives of Dirac measure δ_b . Since the support of $\hat{\mu}_A$ is a discrete set,we conclude that

$$\hat{\mu}_A = \sum_{b \in B} \psi_b$$

where

$$\psi_b = \sum_{j=0}^{m_b} c_{b,j} D^j \delta_b$$

In order to show that $\hat{\mu}_A$ is a measure, we must show that each distribution ψ_b has zero order, i.e $m_b = 0$ for all $b \in B$.

Fix $b \in B$ and take a test function $\phi \in C_c^{\infty}(-1,1)$ such that $D^j\phi(0) = (-1)^j$, for $0 \leq j \leq m_b$. We consider the distribution $\hat{\mu}_A$ acting on the scaled test function $g(x) = \phi(\lambda(x-b))$, whose Fourier Transform is given by

$$\hat{g}(\xi) = \frac{1}{\lambda} e^{-2\pi\xi b/\lambda} \hat{\phi}(\xi/\lambda)$$

If λ is large enough, then

$$\mathrm{supp} g \cap B = \{b\}$$

because $supp\phi \subseteq (-1,1)$. In this case

$$\begin{aligned}
\hat{\mu}_A(g) &= \psi_b(g) \\
&= \sum_{j=0}^{m_b} c_{b,j} D^j \delta_b(g) \\
&= \sum_{j=0}^{m_b} c_{b,j} (-1)^j D^j g(b) \\
&= \sum_{j=0}^{m_b} c_{b,j} (-1)^j \lambda^j D^j \phi(0) \\
&= \sum_{j=0}^{m_b} c_{b,j} \lambda^j
\end{aligned}$$

This is a polynomial in λ of degree m_b. On the other hand,

$$\hat{\mu}_A(g) = \mu_A(\hat{g}) = \sum_{a\in A} \hat{g}(a)$$

We enumerate the points of the tile set A in increasing order of absolute value $A = \{a_n\}_{n\in\mathbb{N}}$ The bounded density of A implies $|a_n| \gg n$. Thus as $n \to \infty$

$$|\hat{g}(a_n)| = \frac{1}{\lambda}|\hat{\phi}(a_n/\lambda)| \ll \frac{1}{\lambda}|\frac{a_n}{\lambda}|^{-3/2} \ll \lambda^{1/2} n^{-3/2}$$

where we used $|\hat{\phi}(\xi)| \ll |\xi|^{-3/2}$. Thus as $\lambda \to \infty$

$$|\sum_{j=0}^{m_b} c_{b,j}\lambda^j| = |\hat{\mu}_A(g)| \ll \lambda^{1/2} \sum_{n=1}^{\infty} n^{-3/2} \ll \lambda^{1/2}$$

This implies $m_b = 0$. Thus $\hat{\mu}_A$ is a Radon measure

$$\hat{\mu}_A = \sum_{b\in B} c_b \delta_b, \quad c_b \in \mathbb{C}. \tag{2.3}$$

Step 2: The coefficients c_b in (2.3) are bounded.
Fix $b \in B$ and take a test function $\phi \in C_c^\infty(-1,1)$ such that $\phi(0) = 1$. We consider the scaled test function $h(x) = \phi(\lambda(x-b))$, whose Fourier Transform is given by

$$\hat{h}(\xi) = \frac{1}{\lambda} e^{-2\pi\xi b/\lambda} \hat{\phi}(\xi/\lambda)$$

If λ is large enough, then

$$supp h \cap B = \{b\}$$

because $supp \phi \subseteq (-1,1)$. We also consider the enumeration of $A = \{a_n\}_{n\in\mathbb{N}}$ as before. We have

$$c_b = \hat{\mu}_A(h) = \mu_A(\hat{h}) = \sum_{n=1}^{\infty} \hat{h}(a_n)$$

hence,

$$|c_b| \leq \frac{1}{\lambda} \sum_{n=1}^{\infty} |\hat{\phi}(a_n/\lambda)| = S_1 + S_2$$

where S_1, S_2 represent the sum taken over $|a_n| \leq \lambda$ and $|a_n| > \lambda$, respectively. Since $|a_n| \gg n$ and using the estimate $|\hat{\phi}(a_n/\lambda)| \leq |\hat{\phi}|_{[-1,1]}$, we obtain

$$S_1 \leq \frac{1}{\lambda}|\hat{\phi}|_{[-1,1]}|\{n : |a_n| \leq \lambda\}| \ll \frac{1}{\lambda}\lambda \ll 1.$$

Similarly using the estimate $|\hat{\phi}(a_n/\lambda)| \ll |a_n/\lambda|^{-2} \ll \lambda^2/n^2$ we bound S_2,

$$S_2 \leq \frac{1}{\lambda}\sum_{|a_n|>\lambda}\frac{\lambda^2}{n^2} \ll \lambda \sum_{n \gg \lambda}\frac{1}{n^2} \ll \lambda\frac{1}{\lambda} \ll 1$$

Thus,

$$|c_b| \ll 1.$$

Step 3: The set A is in the coset ring of $\mathbb{R}$.
The claim is a consequence of Meyer's theorem, since

$$|\hat{\mu}_A|([-R,R]) \leq \sup_{b \in B}|c_b| \cdot |\{b \in B : |b| \leq R\}| \ll R$$

by the assumption on the zero set $B = \{0\} \cup \mathcal{Z}(\hat{f})$. We have that the measure μ_A satisfies the hypotheses of Meyer's theorem, so A is in the coset ring of $\mathbb{R}$ and by Rosenthal's theorem has the form

$$A = F \triangle \bigcup_{j=1}^{k}(\alpha_j\mathbb{Z} + \beta_j),$$

for some finite set F.
Step 4: The finite set F is empty, which proves the theorem.
If $A' = \bigcup_{j=1}^{k}(\alpha_j\mathbb{Z} + \beta_j)$ then the (PSF) implies that $\hat{\mu}_{A'}$ is a weighted sum of point masses on arithmetic progressions. Then

$$\hat{\mu}_A = \hat{\mu}_{A'} + \sum_{n \in F \setminus (A' \cap F)} e^{-2\pi i \xi} - \sum_{n \in (A' \cap F)} e^{-2\pi i \xi}$$

This shows that F must be empty, since $\hat{\mu}_A$ has no continuous parts by (2.3). □

Theorem 2.3.2. *[5]Let a non-zero function $f \in L^1(\mathbb{R})$ have compact support. Then any tile set A of bounded density for f is a finite union of arithmetic progressions*

$$A = \bigcup_{j=1}^{J}(\alpha_j\mathbb{Z} + \beta_j), \quad all \quad \alpha_j > 0.$$

Proof. We show that if $f \in L^1(\mathbb{R})$ has compact support then its Fourier transform $\hat{f}$ satisfies the hypotheses of theorem 6,i.e $\hat{f} \in C^\infty(\mathbb{R})$ which has a discrete zero set satisfying (2.2). If f has support in $[-R, R]$ then the Fourier transform defined on the complex numbers

$$\hat{f}(z) = \int_{-R}^{R} f(z)e^{-ixz}dx, \quad z \in \mathbb{C},$$

is an entire function, which satisfies the growth bound

$$|\hat{f}(z)| = \int_{-R}^{R} e^{xIm(z)}|f(x)|dx \leq \|f\|_1 e^{R|z|}.$$

If $N(T)$ counts the number of zeros of $\hat{f}(z)$ in the disc $\{z : |z| \leq T\}$, an application of Jensen's formula (Boas [8], theorem 2.5.13) gives

$$\limsup_{T\to\infty} \frac{N(T)}{T} \leq eR$$

which implies (2.2).

□

Chapter 3

Structure in dimension 2

3.1 Preliminaries

In this chapter we prove that a class of polygonal regions which tiles the plane $\mathbb{R}^2$ have specific structure. The main reference for this chapter is Kolountzakis [9].

Definition 3.1.1 (Polygons with the pairing property). *A polygon $K \subseteq \mathbb{R}^2$ has the pairing property if for every edge e of K there is precisely one other edge of K parallel to e.*

Remark 3.1.1.
1. *The polygonal regions we deal are not assumed to be connected.*
2. *All symmetric convex polygons have the pairing property and it is easy to see that all convex polygons that tile by translation are necessarily symmetric.*

Definition 3.1.2 (Quasi-Periodic multiset). *A multiset $\Lambda \subseteq \mathbb{R}^d$ is called quasi-periodic if it is a finite union of translated d-dimensional lattices in $\mathbb{R}^d$.*

Suppose K a polygon with the pairing property and e_1, e_2 is a pair of parallel edges in the direction u. Then, as we noticed, e_1, e_2 have the same length.
If μ_u is the measure which is equal to arc-length on e_1 and negative arc-length on e_2 then, if $K + \Lambda$ is a multiple tiling of $\mathbb{R}^2$,

$$\sum_{\lambda \in \Lambda} \mu_u(x - \lambda)$$

is the zero measure, since every part of a translate of e_1 in the tiling has to be cancelled by part of a copy of e_2.
It is also intuitively obvious that the vanishing of the above measure for all directions u implies that $K+\Lambda$ is a multiple tiling of $\mathbb{R}^2$.

So if we write $e_2 = e_1 + \tau$ for some vector $\tau \in \mathbb{R}^2$ (where e_1, e_2 are viewed as point sets), we have made the following observation

Observation 3.1.1. *If a polygon K has the pairing property then it tiles multiply the plane, i.e. $K+\Lambda = w\mathbb{R}^2$, if and only if for each pair e and $e+\tau$ of parallel edges of K,*

$$\sum_{\lambda\in\Lambda} \mu_{e,\tau}(x-\lambda) = 0 \tag{3.1}$$

where $\mu_{e,\tau}$ is the measure which is equal to arc-length on e and negative arc-length on $e+\tau$.

In this point we want again to use harmonic analysis so we rewrite the relation (3.1) of the observation as a convolution

$$\mu_{e,\tau} * \delta_\Lambda = 0$$

Now we notice that, if $K+\Lambda$ is a tiling, then Λ cannot contain more than cR^2 points in any disc of radius $R, R>1$, where c is a constant which depends on K and the weight of the tiling. Hence, we have the growth condition (1.2), which implies that δ_Λ is a tempered distribution and so we can take the Fourier Transform in the convolution above

$$\hat{\mu}_{e,\tau}\hat{\delta_\Lambda} = 0$$

which implies

$$\mathrm{supp}\hat{\delta_\Lambda} \subseteq \mathcal{Z}(\hat{\mu}_{e,\tau}). \tag{3.2}$$

3.2 The Shape of the Zero-Set

In this section we study the shape of the zero-set $\mathcal{Z}(\hat{\mu}_{e,\tau})$ and determine its structure. First we calculate the Fourier transform of $\mu_{e,\tau}$ in the particular case when e is parallel to the x-axis, for simplicity. We define the measure $\mu \in M(\mathbb{R}^2)$ by duality by

$$\mu(\phi) = \int_{-1/2}^{1/2} \phi(x,0)dx, \qquad \phi \in C(\mathbb{R}^2)$$

That is μ is the arc-length on the line-segment joining the points $(-1/2, 0), (1/2, 0)$. A simple calculation using Fubini's theorem gives

$$\hat{\mu}(\xi, \eta) = \frac{sin\pi\xi}{\pi\xi}$$

If μ_L is the arc-length on the line-segment joining the points $(-L/2, 0), (L/2, 0)$ we have

$$\hat{\mu}(\xi, \eta) = \frac{sin\pi L\xi}{\pi\xi}$$

hence

$$\mathcal{Z}(\hat{\mu_L}) = \{(\xi, \eta) : \xi \in L^{-1}\mathbb{Z} \setminus \{0\}\}$$

Write $\tau = (a, b)$ and $\mu_{L,\tau}$ for the arc-length on the line-segment joining the points $(-L/2, 0), (L/2, 0)$ translated by $\tau/2$ and the negative arc-length on the same segment translated by $-\tau/2$. That is, we have

$$\mu_{L,\tau} = \mu_L * (\delta_{\tau/2} - \delta_{-\tau/2})$$

and, taking the Fourier transform, we get

$$\hat{\mu}_{L,\tau} = -2\frac{\sin\pi L\xi}{\pi\xi}\sin(a\xi + b\eta)$$

Hence, if we define $u = \tau/|\tau^2|$ and $v = (1/L, 0)$, we have

$$\mathcal{Z}(\hat{\mu}_{L,\tau}) = (\mathbb{Z}u + \mathbb{R}u^{\perp}) \cup (\mathbb{Z} \setminus \{0\}v + \mathbb{R}v^{\perp})$$

where $u^{\perp}, v^{\perp}$ are the unit vectors orthogonal to u, v respectively.

Therefore the shape of the zero set of the measure $\mu_{L,\tau}$ is a set of straight lines of direction $u^{\perp}$ evenly spaced by $|u|$, plus a similar set of lines of direction $v^{\perp}$ spaced by $|v|$. However in the latter set of parallel lines the straight line through zero has been removed (See Figure 3.1). We state this as a theorem for later use.

Definition 3.2.1 (Geometric Inverse of a vector)**.** *The geometric inverse of a non zero vector $u \in \mathbb{R}^2$ is the vector*

$$u^* = \frac{u}{|u^2|}$$

Theorem 3.2.1. *[9] Let e and $e + \tau$ be two parallel line segments, of magnitude and direction described by e and symmetric with respect to 0. Let also $\mu_{e,\tau}$ be the measure which is equal to arc-length on e and negative arc-length on $e + \tau$. Then*

$$\mathcal{Z}(\hat{\mu}_{e,\tau}) = (\mathbb{Z}\tau^* + \mathbb{R}\tau^{*\perp}) \cup (\mathbb{Z} \setminus \{0\}e^* + \mathbb{R}e^{*\perp}).$$

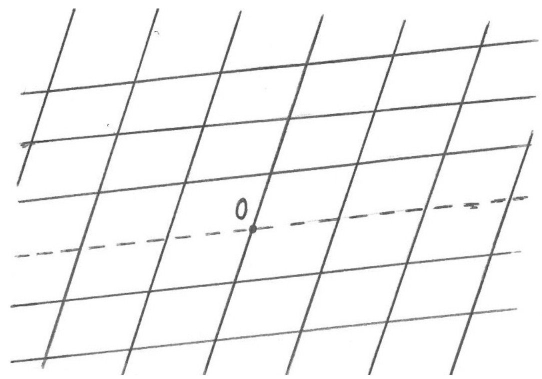

Figure 3.1: The shape of the zero set $\mathcal{Z}(\hat{\mu}_{e,\tau})$.

3.3 Quasi-Periodic Structure in $\mathbb{R}^d$

Theorem 3.3.1. *[9] Suppose that $\Lambda \subseteq \mathbb{R}^d$ is a multiset with density ρ. If the Fourier transform $\hat{\delta}_\Lambda$ of the distribution $\delta_\Lambda = \sum_{\lambda\in\Lambda}\delta_\lambda$ is a measure in a neighbourhood of zero, then $\hat{\delta}_\Lambda(\{0\}) = \rho$.*

Proof. Take a test function $\phi \in C_c^\infty(\mathbb{R}^d)$ with $\phi(0) = 1$. Then we have

$$\begin{aligned}
\hat{\delta}_\Lambda(\{0\}) &= \lim_{t\to\infty} \hat{\delta}_\Lambda(\phi(tx)) \\
&= \lim_{t\to\infty} \delta_\Lambda(t^{-d}\hat{\phi}(\xi/t)) \\
&= \lim_{t\to\infty} t^{-d}\sum_{\lambda\in\Lambda}\hat{\phi}(\lambda/t)) \qquad . \\
&= \lim_{t\to\infty} \sum_{n\in\mathbb{Z}}\ \sum_{\lambda\in Q_n\cap\Lambda} t^{-d}\hat{\phi}(\lambda/t))
\end{aligned}$$

Where, for fixed $T > 0$,

$$Q_n = [0,T)^d + Tn, \qquad n \in \mathbb{Z}.$$

Since Λ has bounded density ρ it follows that for each $\varepsilon > 0$ we can choose T large enough so that, for all n,

$$|\Lambda \cap Q_n| = \rho|Q_n|(1 + \delta_n),$$

with $|\delta_n| \le \varepsilon$.
From the mean value theorem for $\hat{\phi}$, we have that for each n and $\lambda \in Q_n$ we have

$$\hat{\phi}(\lambda/t) = \hat{\phi}(Tn/t) + r_\lambda$$

where $|r_\lambda| \le CTt^{-1} \left\| \nabla \hat{\phi} \right\|_{L^\infty(t^{-1}Q_n)}$.
Hence,

$$\begin{aligned}
\hat{\delta}_\Lambda(\{0\}) &= \lim_{t\to\infty} \sum_{n\in\mathbb{Z}} t^{-d} \sum_{\lambda \in Q_n} (\hat{\phi}(Tn/t) + r_\lambda) \\
&= \lim_{t\to\infty} \sum_{n\in\mathbb{Z}} t^{-d} \rho |Q_n| (1+\delta_n) \hat{\phi}(Tn/t) \\
&+ \lim_{t\to\infty} \sum_{n\in\mathbb{Z}} t^{-d} \sum_{\lambda\in Q_n} r_\lambda \\
&= \lim_{t\to\infty} S_1 + \lim_{t\to\infty} S_2
\end{aligned}$$

For S_1 we have

$$|S_1 - \sum_{n\in\mathbb{Z}} \rho |Q_n| \hat{\phi}(Tn/t)| \le \varepsilon \sum_{n\in\mathbb{Z}} \rho |Q_n| |\hat{\phi}(Tn/t)|,$$

where the first sum $\sum_{n\in\mathbb{Z}} \rho|Q_n|\hat{\phi}(Tn/t)$ is a Riemann sum for $\rho \int_{\mathbb{R}^d} \hat{\phi} = \rho$ and the second sum $\sum_{n\in\mathbb{Z}} \rho|Q_n||\hat{\phi}(Tn/t)|$ is a Riemann sum for $\rho \int_{\mathbb{R}^d} |\hat{\phi}| < \infty$. Therefore, since ε was arbitrary, $\lim_{t\to\infty} S_1 = \rho$.
For S_2 we have

$$\begin{aligned}
|S_2| &\le C \sum_{n\in\mathbb{Z}} t^{-d} \rho |Q_n| (1+\delta_n) Tt^{-1} \left\| \nabla \hat{\phi} \right\|_{L^\infty(t^{-1}Q_n)} \\
&\le C\rho T t^{-1} \sum_{n\in\mathbb{Z}} t^{-d} |Q_n| \left\| \nabla \hat{\phi} \right\|_{L^\infty(t^{-1}Q_n)}
\end{aligned}$$

The last sum above is a Riemann sum for $\int_{\mathbb{R}^d} |\nabla \hat{\phi}|$, which is finite, since $\hat{\phi}$ is in Schwartz space. Hence $\lim_{t\to\infty} S_2 = 0$. □

Remark 3.3.1. *The same proof as Theorem 3.3.1 shows that if*

$$\mu = \sum_{\lambda\in\Lambda} c_\lambda \delta_\lambda, \quad \textit{where} \quad c_\lambda \le C$$

Λ is of zero density, and the tempered distribution $\hat{\mu}$ is locally a measure in a neighbourhood of some $a \in \mathbb{R}^d$ then we have $\mu(\{a\}) = 0$

Theorem 3.3.2. *[9] Suppose that $\Lambda \subseteq \mathbb{R}^2$ is a discrete multiset of uniformly bounded density and that $\hat{\delta}_\Lambda$ is locally a measure with*

$$|\hat{\delta_\Lambda}|(B_R(0)) \ll R^d, \quad as \quad R \longrightarrow \infty.$$

Assume that also $\hat{\delta}_\Lambda$ has discrete support. Then Λ is a finite union of translated lattices.

Proof. Define the sets (not multisets)

$$\Lambda_k = \{\lambda \in \Lambda : \lambda \quad has \quad multiplicity \quad k\}.$$

By Meyer's theorem (applied for the base set Λ with coefficients c_λ equal to the corresponding multiplicities) each of the Λ_k is in the coset ring of $\mathbb{R}^2$ and from Theorem 1.3.4 because it is discrete it is of the type

$$\bigcup_{j=1}^{J}(A_j \setminus (B_1^j \cup ... \cup B_n^j)) \cup \bigcup_{l=1}^{L} L_l \triangle F$$

where A_j are two-dimensional translated lattices with pairwise intersections which have dimension at most one, B_i^j, L_l are one-dimensional translated lattices and $F \subseteq \mathbb{R}^2$ is a finite set.
We may write

$$\Lambda_k = A \triangle B,$$

with $A = \cup_{j=1}^{J} A_j$, where A_j are 2-dimensional translated lattices which have pairwise intersection at most one, and $\mathrm{dens}\,\mathrm{B} = 0$.
Hence

$$\delta_{\Lambda_k} = \sum_{j=1}^{J} \delta_{A_j} + \mu,$$

where $\mu = \sum_{f \in F} c_f \delta_f$, with $\mathrm{dens}\,\mathrm{F} = 0$ and $|c_f| \leq C(J)$. The set F consists of B and all points contained in at least two of the A_j.
Combining for all k, and reusing the symbols A_j, μ, F we have

$$\delta_\Lambda = \sum_{j=1}^{J} \delta_{A_j} + \mu.$$

However, $\hat{\delta}_\Lambda$ is discrete by assumption and $\sum_{j=1}^{J} \hat{\delta}_{A_j}$ is also discrete by PSF. This implies that $\hat{\mu}$ is discrete. However from Remark 3.3.1, since $\mathrm{dens}\,\mathrm{F} = 0$

and the coefficients c_f are bounded, we have that $\hat{\mu}$ has no point masses which means that $\hat{\mu} = 0$. Now, because the Fourier Transform is an automorphism in the space of tempered distributions we conclude that $\mu = 0$.
Hence

$$\delta_\Lambda = \sum_{j=1}^{J} \delta_{A_j} \quad \Leftrightarrow \quad \Lambda = \bigcup_{j=1}^{J} A_j,$$

where the last equality is between multisets. □

The next theorem shows that the discrete support of $\hat{\delta}_\Lambda$ implies that $\hat{\delta}_\Lambda$ is locally a measure.

Theorem 3.3.3. *[9] Suppose that $\Lambda \subseteq \mathbb{R}^d$ is a multiset with density uniformly bounded by ρ and that for some $a \in \mathbb{R}^d$ and $R > 0$,*

$$\mathrm{supp}\hat{\delta}_\Lambda \cap B_R(a) = \{a\}.$$

Then in $B_R(a)$, we have $\hat{\delta}_\Lambda = w\delta_a$, for some $w \in \mathbb{C}$ with $|w| \leq \rho$

Proof. We use again the fact that the tempered distributions supported at a point a are finite combination of derivatives of δ_a. So, for $\phi \in C^\infty(B_R(a))$, we have

$$\hat{\delta}_\Lambda(\phi) = \sum_a c_a (D^a \delta_a)(\phi) = \sum_a c_a (-1)^{|a|} D^a \phi(a), \tag{3.3}$$

where the sum extends over all values of the multiindex $\alpha = (\alpha_1 + ... + \alpha_d)$, with $|\alpha| = \alpha_1 + ... + \alpha_d \leq m$. We want to show that $m = 0$. Assume the contrary and let α_0 be a multiindex with a non-zero coefficient and $|\alpha_0| = m$. Pick a smooth function ϕ supported at a neighbourhood of 0 such that for each multiindex α with $|\alpha| \leq m$ we have $D^\alpha \phi(0) = 0$ if $\alpha \neq \alpha_0$ and $D^{\alpha_0}\phi(0) = 1$.
Let $\phi_t(x) = \phi(t(x - \alpha))$. Then, from (2.3) we have that

$$\hat{\delta}_\Lambda(\phi_t) = t^m (-1)^m c_{\alpha_0}, \tag{3.4}$$

and on the other hand, using the Fourier transform, we have

$$\phi(t(x - \alpha))\widehat{\ }(\xi) = e^{-2\pi i \langle \alpha, \xi/t \rangle} t^{-d} \hat{\phi}(\xi/t),$$

and we get

$$\hat{\delta}_\Lambda(\phi_t) = \sum_{\lambda \in \Lambda} e^{-2\pi i \langle \alpha, \lambda/t \rangle} t^{-d} \hat{\phi}(\lambda/t). \tag{3.5}$$

By a proof similar to that of Theorem 3.3.1 we get that (3.5) is a bounded quantity as $t \to \infty$, while (3.4) increases like t^m, a contradiction. □

The following theorem which is the main result of this section is a consequence of the previous theorems and gives structure for the multiset $\Lambda \subseteq \mathbb{R}^d$.

Theorem 3.3.4 (Kolountzakis)**.** *[9] Suppose that for a multiset* $\Lambda \subseteq \mathbb{R}^d$

1. Λ *has uniformly bounded density,*

2. $\hat{\delta}_\Lambda$ *is a discrete measure,*

3. $|\hat{\delta_\Lambda}|(B_R(0)) \ll R^d, \quad as \quad R \longrightarrow \infty.$

Then Λ *is quasi-periodic, i.e. a finite union of translated d-dimensional lattices.*

3.4 Application to tilings by polygons

In this section we apply Theorem 3.3.4 and use the shape of the zero set $\mathcal{Z}(\hat{\mu}_{e,\tau})$, which we described explicitly in section 3.2, in order to give quasi-periodic structure for the tilings formed by a class of polygons.

Theorem 3.4.1. *[9] Let the polygon* K *have the pairing property and tile multiply the plane with the multiset* Λ*. Denote the edges of* K *by*

$$e_1, e_1 + \tau_1, e_2, e_2 + \tau_2, ..., e_n, e_n + \tau_n.$$

Suppose also that

$$\{\bar{e_1}, \bar{\tau_1}\} \cap ... \cap \{\bar{e_n}, \bar{\tau_n}\} = \emptyset, \tag{3.6}$$

where $\bar{v}$ *denotes the orientation of the vector* v*. Then* Λ *is a finite union of 2-dimensional translated lattices.*

Proof. By Observation 2 (3.1) and (3.2), the tiling assumption implies

$$\mathrm{supp}\hat{\delta}_\Lambda \subseteq \mathcal{Z}(\hat{\mu}_{e_1,\tau_1}) \cap ... \cap \mathcal{Z}(\hat{\mu}_{e_n,\tau_n}).$$

By Theorem 3.2.1 in the intersection above each of the sets is contained in a collection of lines in the direction $\bar{e}_i$ union a collection of lines in the direction of $\bar{\tau}_i$. Hence the intersection is a discrete set unless an entire line lies in it, but this case excluded by assumption (3.6). Furthermore, because of the regular spacing of these pairs of sets of lines, it follows that the resulting intersection has at most CR^2 points in a disc of radius R. Theorem 3.3.4 now implies that Λ is a finite union of 2-dimensional translated lattices. $\square$

Theorem 3.4.2. *[9] Suppose that K is a symmetric convex polygon which is not a parallelogram. Then K admits only quasi-periodic multiple tilings.*

Proof. Suppose that (3.6) fails and that the intersection contains a vector which is, say, parallel to the x-axis. It follows that each pair of edges $e_i, e_i+\tau_i$ of edges of K either has both edges parallel to x-axis, or has the line joining the two midpoints parallel to the x-axis. In both cases this can only happen for one pair of edges, which means that K is a parallelogram. □

Remark 3.4.1. *It is clear that parallelograms admit tilings which are not quasi-periodic. We can take the regular tiling by a square, i.e. with lattice $\mathbb{Z}^2$, and move each vertical column of squares arbitrarily up or down.*

Chapter 4

Structure in dimension 3

4.1 Preliminaries

Definition 4.1.1 (Minkowski sum and Zonotopes)**.** *The Minkowski sum of the polytopes $P_1, P_2, ..., P_n \subseteq \mathbb{R}^d$ is defined as*

$$\mathcal{M}(P_1, P_2, ..., P_n) = P_1 + P_2 + ... + P_n = \{x_1 + x_2 + ... + x_n : x_j \in P_j\}.$$

A zonotope is the Minkowski sum of a finite number of line segments.

Suppose that we are now given n line segments in $\mathbb{R}^d$, such that each line segment has one endpoint at the origin and the other endpoint is located at the vector $u_j \in \mathbb{R}^d$. Then by definition, the Minkowski sum of these n segments is

$$\mathcal{M}(u_1, u_2, ..., u_n) = \{\lambda_1 u_1 + \lambda_2 u_2 + ... + \lambda_n u_n : \lambda_j \in [0, 1]\} = A[0, 1]^d,$$

where A is the $(d \times n)$-matrix whose j-th column is u_j, i.e. a zonotope is a translate of $A[0, 1]^d$. Hence, a zonotope may equivalently be defined as the projection of some l-dimensional cube.
It is sometimes useful to translate a zonotope so that the origin becomes its new center of mass. To this end, we now dilate the matrix A by a factor of 2 and translate the resulting image so that its new center of mass is at the origin. So we can redefine a zonotope as

$$\mathcal{M}(u_1, u_2, ..., u_n) = A[-1, 1]^d,$$

which shows that a zonotope is centrally symmetric. It is moreover true that each face of a zonotope is again a zonotope and that therefore every face of

a zonotope is centrally symmetric.
Another fact that we will use is that any zonotope can be decomposed into a disjoint union of half-open parallelepipeds. This can be proved by induction on the number of line segments in Minkowski sum.

For any symmetric polytope P, and any face $F \subseteq P$, we define F^- to be the face opposite of F with respect to $P's$ center of symmetry. Also we denote $\mathbf{x}^\perp$ the perpendicular subspace to the vector $\mathbf{x}$. We also use the standard convention of boldfacing all vectors. We furthermore use the convention that $[\mathbf{e}]$ denotes the 1-dimensional line segment from 0 to the endpoint of the vector $\mathbf{e}$.

Definition 4.1.2 (4-legged-frame of a polytope)**.** 1. *Suppose $P \in \mathbb{R}^3$ is a zonotope. A collection of four, one-dimensional, edges of P is called a 4-legged-frame if whenever e is one of the edges then there exist two vectors τ_1 and τ_2 such that the four edges are*

$$[e], [e] + \tau_1, [e] + \tau_2, [e] + \tau_1 + \tau_2$$

and such that the edges $[e]$, $[e] + \tau_1$ belong to the same face of P and the edges $[e] + \tau_2$ and $[e] + \tau_1 + \tau_2$ belong to the opposite face.

2. *For a set of four legs as above the leg measure is the measure supported on the legs and is equal to arc-length on the two legs $[e]$ and $[e]+\tau_1+\tau_2$ and minus arc-length on the two legs $[e] + \tau_1$ and $[e] + \tau_2$. We denote this measure by μ_{e,τ_1,τ_2}.*

4.2 The FT of a 4-legged-frame

Lemma 4.2.1. *[10] Suppose $\mathbf{e}, \tau_1, \tau_2 \in \mathbb{R}^3$ are linearly independent and consider the leg measure $\mu = \mu_{e,\tau_1,\tau_2}$. Then the zero-set of the Fourier Transform $\hat{\mu}$, is*

$$\mathcal{Z}(\hat{\mu}) = \mathbb{H}_{-0}(\mathbf{e}) \cup \mathbb{H}(\tau_1) \cup \mathbb{H}(\tau_2), \tag{4.1}$$

where

$$\mathbb{H}(x) = \mathbb{Z}\mathbf{x}^* + \mathbf{x}^\perp$$

and

$$\mathbb{H}_{-0}(\mathbf{x}) = (\mathbb{Z} \setminus \{0\})\mathbf{x}^* + \mathbf{x}^\perp$$

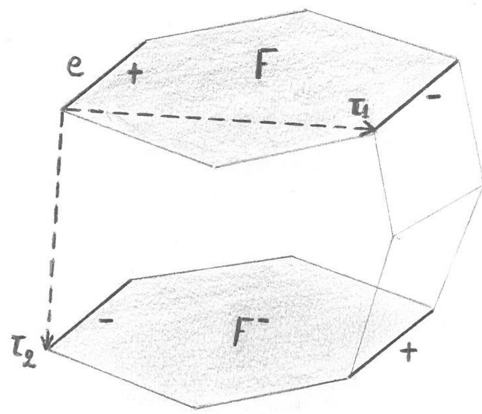

Figure 4.1: A 4-legged-frame of a convex polytope.

Where $\mathbf{x}$ *is a non-zero vector and* $\mathbf{x}^*$ *is the geometric inverse of* $\mathbf{x}$*. Here* $\mathbf{x}^\perp$ *is the plane orthogonal to the vector* $\mathbf{x}$*, so that* $\mathbb{H}(x) = \mathbb{Z}\mathbf{x}^* + \mathbf{x}^\perp$ *is a collection of parallel planes, orthogonal to* $\mathbf{x}$ *spaced by* $1/|\mathbf{x}|$*.*

Proof. Translating a measure doesn't affect the zero set of its FT so we can translate μ so that 0 is the midpoint of the first line segment, which now runs from $-\mathbf{e}/2$ to $\mathbf{e}/2$. Denoting by ν the arc-length measure on this line segment and writing $\alpha = \delta_0 - \delta_{\tau_1}$ and $\beta = \delta_0 - \delta_{\tau_2}$ we obtain μ as convolution:

$$\mu = \nu * \alpha * \beta.$$

Taking the FT we get that

$$\mathcal{Z}(\hat{\mu}) = \mathcal{Z}(\hat{\nu}) \cup \mathcal{Z}(\hat{\alpha}) \cup \mathcal{Z}(\hat{\beta})$$

Based on calculation of the FT of the indicator function of $[-1/2, 1/2]$

$$\int_{-1/2}^{1/2} e^{-2\pi i \xi x} dx = \frac{\sin \pi \xi}{\pi \xi}$$

we conclude that

$$\hat{\nu}(\mathbf{u}) = |\mathbf{e}| \frac{\sin \pi \langle \mathbf{u}, \mathbf{e} \rangle}{\langle \mathbf{u}, \mathbf{e} \rangle}.$$

One also immediately obtains the formulas

$$\hat{\alpha}(\mathbf{u}) = 2ie^{-\pi i \langle \tau_1, \mathbf{u} \rangle} \sin \pi \langle \tau_1, \mathbf{u} \rangle$$

and

$$\hat{\beta}(\mathbf{u}) = 2ie^{-\pi i\langle\tau_2,\mathbf{u}\rangle}\sin\pi\langle\tau_2,\mathbf{u}\rangle.$$

Now we notice that $\hat{\nu}, \hat{\alpha}, \hat{\beta}$ vanish precisely on $\mathbb{H}_{-0}(\mathbf{e}), \mathbb{H}(\tau_1), \mathbb{H}(\tau_2)$, respectively.

□

4.3 Application to tilings by convex polytopes

In this section we use again Theorem 3.3.4 to obtain a quasi-periodic structure for a class of convex polytopes.

Theorem 4.3.1. *[10] Suppose P is a symmetric polytope in $\mathbb{R}^3$ and $\Lambda \subseteq \mathbb{R}^3$ is a multiset such that, $P + \Lambda = k\mathbb{R}^3$, is a k-tiling of $\mathbb{R}^3$. Then we have*

$$\mathrm{supp}\hat{\delta_\Lambda} \subseteq \{0\} \cup \bigcap_{e,\tau_1,\tau_2} (\mathbb{H}_{-0}(\mathbf{e}) \cup \mathbb{H}(\tau_1) \cup \mathbb{H}(\tau_2)), \tag{4.2}$$

where the intersection above is taken over all 4-legged frames (e, τ_1, τ_2) of P.

Proof. We have from [11] (Lemma 3.1 and Lemma 3.2) that if $P + \Lambda$ is k-tiling of $\mathbb{R}^3$ and μ is a leg measure on P then μ also tiles with Λ with zero weight, i.e. $\mu * \delta_\Lambda = 0$. Since $P + \Lambda = k\mathbb{R}^3$, is a k-tiling of $\mathbb{R}^3$ it follows that $|\Lambda \cap B_R(0)| \ll R^3, \quad (as \quad R \to \infty)$. Hence δ_Λ is a tempered distribution and we may take its FT which gives us

$$\hat{\mu}\hat{\delta}_\Lambda = 0,$$

this implies

$$\mathrm{supp}\hat{\delta_\Lambda} \subseteq \{0\} \cup \mathcal{Z}(\hat{\mu}).$$

But the zero set of $\hat{\mu}$ is described in (4.1) at Lemma 4.2.1 and since this must be true for all sets of four legs of P we conclude (4.2). □

Corollary 4.3.2. *[10] Suppose P is a symmetric polytope in $\mathbb{R}^3$ and $\Lambda \subseteq \mathbb{R}^3$ is a multiset such that, $P + \Lambda = k\mathbb{R}^3$, is a k-tiling of $\mathbb{R}^3$ and let the following intersection property hold:*

$$\bigcap_{e,\tau_1,\tau_2} (\mathbf{e}^\perp \cup \tau_1^\perp \cup \tau_2^\perp) = \{0\}, \tag{4.3}$$

where the intersection above is taken over all 4-legged frames (e, τ_1, τ_2) of P. Then $\mathrm{supp}\hat{\delta_\Lambda}$ is a discrete set in $\mathbb{R}^3$, of bounded density.

Proof. The sets in the intersection (4.2) are all unions of planes. For this set to be non-discrete it must be the case that it contains an entire line of direction, say a non-zero vector $\mathbf{u} \in \mathbb{R}^3$. This in turn implies that there is a selection $\mathbf{X}_l$ of $\mathbf{e}, \tau_1, \tau_2$ for each set l of four legs such that $\mathbf{u} \in \mathbf{X}_l^\perp$, but this contradicts (4.3).
Having established that the intersection in (4.2) is a discrete set we observe that the larger set

$$\bigcap_{e,\tau_1,\tau_2} \mathbb{H}(\mathbf{e}) \cup \mathbb{H}(\tau_1) \cup \mathbb{H}(\tau_2)). \tag{4.4}$$

is a finite union of discrete groups, each of them of the form

$$\bigcap_l \mathbb{H}(l),$$

where l runs over all possible sets of four legs of P and for each $l = (e, \tau_1, \tau_2)$ the set $\mathbb{H}(l)$ is one of the $\mathbb{H}(\mathbf{e}), \mathbb{H}(\tau_1), \mathbb{H}(\tau_2)$. Since each discrete group has bounded density so has the the set (4.6), we conclude that supp$\hat{\delta_\Lambda}$ is a discrete set, of bounded density, as its subset. □

Theorem 4.3.3. *[10] Suppose P is a symmetric polytope in $\mathbb{R}^3$ and $\Lambda \subseteq \mathbb{R}^3$ is a multiset such that, $P + \Lambda = k\mathbb{R}^3$, is k-tiling of $\mathbb{R}^3$ and let the intersection property (4.3) of Corollary 4.3.2 hold. Then Λ is quasi-periodic.*

Proof. We need to verify conditions (1),(2) and (3) of Theorem 3.3.4. Hypothesis (1) follows from the fact that in each sufficiently large ball $B_R(x)$ every point is covered exactly k times by the translations of P with the set $\Lambda \cap B_{R'}(x)$ where $R' = R + \text{diamP}$. Hypotheses (2) and (3) follow from the previous corollary. □

4.4 Tilings with non-discrete supp$\hat{\delta_\Lambda}$

In this section we study the convex polytopes that admit exceptional multiple tilings, in the sense that the multiset is not a finite union of 3-dimensional translated lattices. A class of these exceptions is easily provided by prisms (a Minkowski sum of a symmetric polygon with a line segment which doesn't lie in the polygon's plane). By Corollary 4.3.2 for such a tiling the intersection property (4.3) cannot be true. Therefore, there is a line $l \subseteq \mathbb{R}^3$ such that

$$l \subseteq \bigcap_{e,\tau_1,\tau_2} (\mathbf{e}^\perp \cup \tau_1^\perp \cup \tau_2^\perp). \tag{4.5}$$

It was shown in [11] that multiple translational tiling in $\mathbb{R}^3$ must be a zonotope. Here we will show that non-discreteness of supp$\hat{\delta_\Lambda}$ implies that a zonotope is Minkowski sum of two 2-dimensional symmetric polygons.

Definition 4.4.1 (Two-flat zonotope)**.** *We say that a polytope P is a two-flat zonotope in $\mathbb{R}^3$ if P is the Minkowski sum of line segments*

$$[\mathbf{v}_1], ..., [\mathbf{v}_n], [\mathbf{w}_1], ..., [\mathbf{w}_n]$$

where $\mathbf{v}_1, ..., \mathbf{v}_n \in H_1$ and $\mathbf{w}_1, ..., \mathbf{w}_m \in H_2$ and H_1, H_2 are two different 2-dimensional subspaces.

Theorem 4.4.1. *[10] Suppose a polytope P tiles $\mathbb{R}^3$ by translations over a multiset Λ and condition (4.5) holds. Then P is a two-flat zonotope.*

Proof. We let L be a plane orthogonal to l and supporting P, then (4.5)

$$L \perp l \subseteq \bigcap_{e,\tau_1,\tau_2} (\mathbf{e}^\perp \cup \tau_1^\perp \cup \tau_2^\perp), \tag{4.6}$$

implies

$$\forall \mathbf{e}, \tau_1, \tau_2, \quad \textit{either} \quad \mathbf{e} \| L \quad \textit{or} \quad \tau_1 \| L \quad \textit{or} \quad \tau_2 \| L.$$

Let $F = L \cap P$. The dimension of the face F can be 0, 1, or 2. Consider any facet (edge or face) G of P such that:

(i) has at least one common vertex with F,

(ii) and we choose an edge e of G that shares exactly one vertex v with F (so $G \neq F$).

Consider the 4-legged frame determined by G and e with τ_1, τ_2 being the corresponding translation vectors. Since $v \in L$, by (4.6) one of the three vertices $v + \mathbf{e}, v + \tau_1, v + \tau_2$ lies in L, and therefore lies also in F. By our choice of $\mathbf{e}$, $v + \mathbf{e}$ is a vertex of G but not of F, due to (ii). Thus either $v + \tau_1 \in F$ or $v + \tau_2 \in F$

1. If $v + \tau_1 \in F$, then $\tau_1 \in G \cap F$ (since $v, v + \tau_1 \in G \cap F$), so we see that τ_1 is an edge of G. Hence, by the definition of a 4-legged frame, G is a parallelogram.

2. If $v + \tau_2 \in F$, then F connects G with its opposite face G^-See Figure 4.2).

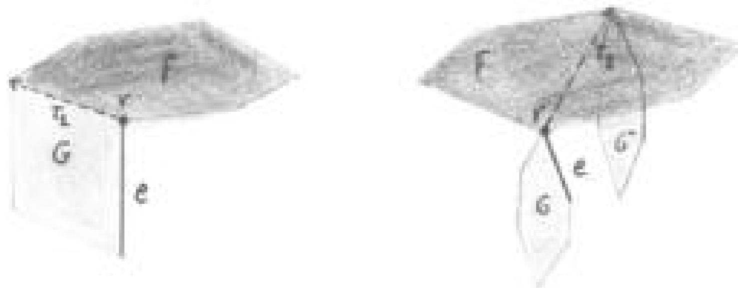

Figure 4.2: The two possibilities for the facet G with respect to F.

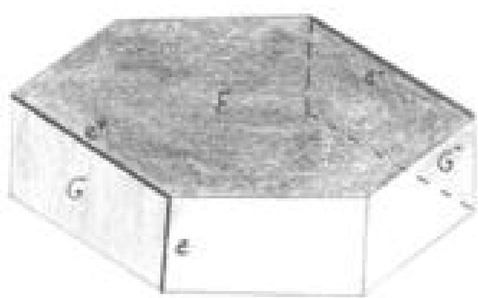

Figure 4.3: This is the case that each facet adjacent to F, is a parallelogram sharing an edge with F, giving us a prism.

Now, we show that there is a facet G which satisfies property 2. If, to the contrary, every facet adjacent to F satisfies property 1, then each facet adjacent to F is a parallelogram sharing an edge with F. It follows that exactly three edges meet at every vertex of F and all edges of these parallelograms that are not edges of F or parallel to F, are parallel to each other. Since F is centrally symmetric, consider two parallel edges e^+ and e^- of F and the corresponding parallelograms G and G^-. The facets G and G^- are parallel and therefore opposite in P, so G satisfies property 2.

Now that we have found a facet G such that F connects G and G^-, we also note that since P is centrally symmetric, G also connects F and F^-. We will show that $P = F + G$. Without loss of generality we may assume that F and G do not share an edge. The case that F and G do in fact share

an edge my be handled in exactly the same manner.
Since P is a zonotope we have

$$P = F + G + H,$$

where H is a polytope and

$$F = [f_1] + ... + [f_k]$$

$$G = [g_1] + ... + [g_l]$$

$$H = [h_0] + [g_1] + ... + [h_m]$$

where $k \geq 0, l \geq 2$, and $m \geq 0$. To arrive at a contradiction we assume that H is not a single point and let h_0 be an edge of H. We may also assume that all line segments have their origin as their midpoint and thus the center of P is at the origin. We further consider a normal vector $\mathbf{f}_\perp$ to the face F of P. When F is 2-dimensional face, $\mathbf{f}_\perp$ cannot be orthogonal to any line segment $h_i \in H$ and $g_i \in G$. If F is a 0 or 1 dimensional face of P we have an infinite collection of perpendicular vectors to F and we may choose $\mathbf{f}_\perp$ to be not orthogonal to any line segment $h_i \in H$ and $g_i \in G$.
For each edge g_i, h_i we define $\mathbf{g}_i^+, \mathbf{g}_i^-$ and $\mathbf{h}_i^+, \mathbf{h}_i^-$, respectively, to be the vector from the origin to the endpoint of g_i, h_i such that

$$\langle \mathbf{g}_i^+, \mathbf{f}_\perp \rangle > 0, \quad \langle \mathbf{g}_i^+, \mathbf{f}_\perp \rangle < 0,$$

$$\langle \mathbf{h}_i^+, \mathbf{f}_\perp \rangle > 0, \quad \langle \mathbf{h}_i^+, \mathbf{f}_\perp \rangle < 0.$$

Now the location of the faces F, F^- in $\mathbb{R}^3$ is given by

$$[f_1] + ... + [f_k] + \mathbf{g}_1^+ + ... + \mathbf{g}_l^+ + \mathbf{h}_1^+ + ... + \mathbf{h}_m^+$$

$$[f_1] + ... + [f_k] + \mathbf{g}_1^- + ... + \mathbf{g}_l^- + \mathbf{h}_1^- + ... + \mathbf{h}_m^-$$

as a set of extremal points, respectively.
Therefore the distance between F and F^- is

$$\mathrm{dist(F, F^-)} = \langle \mathbf{f}_\perp, \sum_{\mathrm{i=0}}^{\mathrm{l}} \mathbf{g}_{\mathrm{i}}^+ + \sum_{\mathrm{i=0}}^{\mathrm{m}} \mathbf{h}_{\mathrm{i}}^+ - \sum_{\mathrm{i=0}}^{\mathrm{l}} \mathbf{g}_{\mathrm{i}}^- - \sum_{\mathrm{i=0}}^{\mathrm{m}} \mathbf{h}_{\mathrm{i}}^- \rangle$$

$$> \langle \mathbf{f}_\perp, \sum_{i=0}^{l} (\mathbf{g}_i^+ - \mathbf{g}_i^-) \rangle$$

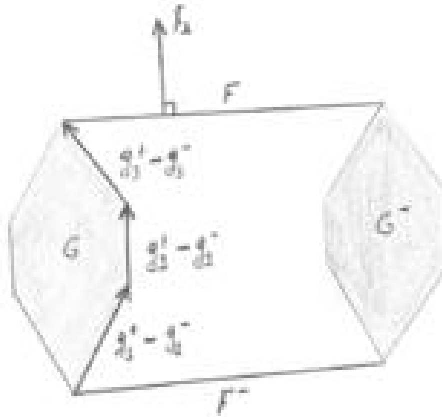

Figure 4.4: Here F is a lower-dimensional face of P, an edge of P, and we see how we can get from the face F to the face F^- by walking along the vectors $\mathbf{g}_i^+ - \mathbf{g}_i^-$.

where the strict inequality follows from the assumption that H contains the edge h_0.

On the other hand, since G connects F and F^-, we have that $F = F^- + \sum_{i\in I}(g_i^+ - \mathbf{g}_i^-)$, for a set I of edges in G. Therefore the distance between F and F^- is not more than

$$\langle \mathbf{f}_\perp, \sum_{i\in I}(\mathbf{g}_i^+ - \mathbf{g}_i^-)\rangle,$$

a contradiction. □

The next theorem which is the main result of this chapter is an immediate consequence of the previous statements.

Theorem 4.4.2. *[10] Suppose a polytope P tiles $\mathbb{R}^3$ by translations over a multiset Λ, and suppose that P is not a two-flat zonotope. Then Λ is a finite union of translated lattices.*

Proof. If P is not a two-flat zonotope then theorem 18 implies that condition (4.5) fails. Therefore, the intersection property (4.3) in corollary 16 holds. Theorem 17 now concludes the proof. □

4.5 Weird tilings

In this section we prove that there are two-flat zonotopes which admit tilings which are not quasi-periodic.

Theorem 4.5.1. *[10] Suppose that P is a two-flat zonotope in $\mathbb{R}^3$, which is the Minkowski sum of the segments*

$$[\mathbf{v}_1], ..., [\mathbf{v}_n], [\mathbf{w}_1], ..., [\mathbf{w}_n],$$

where $\mathbf{v}_1, ..., \mathbf{v}_n \in H_1$ and $\mathbf{w}_1, ..., \mathbf{w}_m \in H_2$ and H_1, H_2 are two different 2-dimensional subspaces. Suppose also that the additive group generated by $\mathbf{v}_1, ..., \mathbf{v}_n, \mathbf{w}_1, ..., \mathbf{w}_m$ is discrete and that the $\mathbf{v}_j$ span H_1. Then P admits a tiling by translations at a set $\Lambda \subseteq \mathbb{R}^3$ which is not a finite union of translated lattices.

Proof. We have that P can be paved by parallelepipeds, whose sides are among the vectors $\mathbf{v}_j$ and $\mathbf{w}_j$. Therefore we can write its indicator function as a finite sum of indicator functions of parallelepipeds.

$$\mathbf{1}_P(x) = \sum_{j=1}^{M} \mathbf{1}_{B_j}(x), \quad a.e.,$$

where each B_j is a parallelepiped whose three edges are among the $\mathbf{v}_j$ and $\mathbf{w}_j$.
Suppose now that the parallelepiped B is centred at the origin and has as edges the three linearly-independent vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$. We can write the indicator function of B convolution

$$\mathbf{1}_B = \frac{|\det(\mathbf{a}, \mathbf{b}, \mathbf{c})|}{|\mathbf{a}| \cdot |\mathbf{b}| \cdot |\mathbf{c}|} \mu_a * \mu_b * \mu_c,$$

where μ_a is the measure that equals arc-length on the line segment from $-\mathbf{a}/2$ to $\mathbf{a}/2$ and μ_b, μ_c are defined similarly. Since we have computed in section 4.2

$$\hat{\mu}_a(\xi) = |\mathbf{a}| \frac{\sin \pi\langle\xi, \mathbf{a}\rangle}{\pi\langle\xi, \mathbf{a}\rangle}$$

and similarly for $\hat{\mu}_b, \hat{\mu}_c$ we obtain the formula

$$\hat{\mathbf{1}}_B(\xi) = |\det(\mathbf{a}, \mathbf{b}, \mathbf{c})| \frac{\sin \pi\langle\xi, \mathbf{a}\rangle}{\pi\langle\xi, \mathbf{a}\rangle} \frac{\sin \pi\langle\xi, \mathbf{b}\rangle}{\pi\langle\xi, \mathbf{b}\rangle} \frac{\sin \pi\langle\xi, \mathbf{b}\rangle}{\pi\langle\xi, \mathbf{b}\rangle}. \tag{4.7}$$

Each parallelepiped B_j in the decomposition of P is a translate of a parallelepiped of the type B, above, with some the vectors $\mathbf{v}_j, \mathbf{w}_j$ in place of $\mathbf{a}, \mathbf{b}, \mathbf{c}$. Translates don't affect the zero set of Fourier Transform, hence

$$\mathcal{Z}(\hat{\mathbf{1}}_{B_j}) = ((\mathbb{Z} \setminus \{0\})\mathbf{a}^* + \mathbf{a}^\perp) \cup ((\mathbb{Z} \setminus \{0\})\mathbf{b}^* + \mathbf{b}^\perp) \cup ((\mathbb{Z} \setminus \{0\})\mathbf{c}^* + \mathbf{c}^\perp),$$

where $\mathbf{a}^*$ is the geometric inverse of $\mathbf{a}$, etc.
Write now

$$G = \langle \mathbf{v}_1, ..., \mathbf{v}_n \rangle$$

for the additive subgroup (lattice) of H_1 generated by $\mathbf{v}_j$'s and $G^* \subseteq H_1$ for its dual group in H_1, i.e.

$$G^* = \{\mathbf{u} \in H_1 : \forall g \in G \quad \langle \mathbf{u}, g \rangle \in \mathbb{Z}\}$$

We claim now that for each $j = 1, ..., n$,

$$H_1^\perp + (G^* \setminus (\mathbf{v}_1^\perp \cup ... \cup \mathbf{v}_n^\perp)) \subseteq \mathcal{Z}(\hat{\mathbf{1}}_{B_j}). \tag{4.8}$$

This follows since at least one side of B_j is equal to a vector $\mathbf{v}_j$ which makes the corresponding factor in (4.7) vanish on any element of G^*, since $\langle \xi, \mathbf{v}_j \rangle \in \mathbb{Z}, \forall \xi \in G^*$, except from those which are orthogonal to $\mathbf{v}_j$, due to denominator. And since that factor in (4.7) is constant along $H_1^\perp$ we obtain the claim. Since (4.8) holds for all j we obtain

$$H_1^\perp + (G^* \setminus (\mathbf{v}_1^\perp \cup ... \cup \mathbf{v}_n^\perp)) \subseteq \mathcal{Z}(\hat{\mathbf{1}}_P). \tag{4.9}$$

Pick now any non-zero $c_1, ..., c_n \in \mathbb{R}$. We claim that

$$\tau = \mathbf{1}_P * \delta_G * (\delta_0 - \delta_{c_1 \mathbf{v}_1}) * ... * (\delta_0 - \delta_{c_n \mathbf{v}_n}) = 0, \tag{4.10}$$

where $\delta_G = \sum_{g \in G} \delta_g$.
For this it is enough to show that the Fourier Transform of the above measure

$$\hat{\tau}(\xi) = \hat{\mathbf{1}}_P(\xi)(1 - e^{2\pi i c_1 \langle \mathbf{v}_1, \xi \rangle})...(1 - e^{2\pi i c_n \langle \mathbf{v}_n, \xi \rangle})\hat{\delta}_G$$

is identically zero.
By the Poisson Summation Formula, it follows that $\hat{\delta}_G$ is a measure with

$$\mathrm{supp}\hat{\delta}_G = G^* + H_1^\perp.$$

By (4.9) the function $\hat{\mathbf{1}}_P(\xi)$ kills $\hat{\delta}_G$ except at the lines of the form $\mathbf{g}^* + H_1^\perp$ with $\mathbf{g}^* \in G^*$ is orthogonal to some $\mathbf{v}_j$, but at these lines one of the factors

$$(1 - e^{2\pi i c_1 \langle \mathbf{v}_1, \xi \rangle}), ..., (1 - e^{2\pi i c_n \langle \mathbf{v}_n, \xi \rangle})$$

vanishes. Hence, $\hat{\tau} = 0$.
Now we rewrite the measure $(\delta_0 - \delta_{c_1 \mathbf{v}_1}) * ... * (\delta_0 - \delta_{c_n \mathbf{v}_n})$ in the form

$$\sum_{k=1}^N \delta_{\mathbf{u}_k^+} - \sum_{k=1}^N \delta_{\mathbf{u}_k^-}, \quad N = 2^{n-1}.$$

Equivalently, we can rewrite (4.10) as the equality

$$\mathbf{1}_P * \delta_G * \sum_{k=1}^N \delta_{\mathbf{u}_k^+} = \mathbf{1}_P * \delta_G * \sum_{k=1}^N \delta_{\mathbf{u}_k^-}. \tag{4.11}$$

We define the multisets

$$S = G + \{\mathbf{u}_1^+, ..., \mathbf{u}_N^+\}, \quad T = G + \{\mathbf{u}_1^-, ..., \mathbf{u}_N^-\}$$

whose ground sets are the supports of the discrete measures

$$\delta_G * \sum_{k=1}^N \delta_{\mathbf{u}_k^+} \quad \textit{and} \quad \delta_G * \sum_{k=1}^N \delta_{\mathbf{u}_k^-}$$

and their multiplicities at each point are those described by these measures. In what follows we exploit (4.11) to give an example of a multiple tiling by P with a discrete set Λ, which cannot be expressed as finite union of translated lattices.
We notice first that since P is a zonotope decomposing into parallelepipeds B_j of sides among the vectors $\mathbf{v}_j, \mathbf{w}_j$, it is a k-tiling of $\mathbb{R}^3$, for some k, with the lattice

$$\Gamma = \langle \mathbf{v}_1, ..., \mathbf{v}_n, \mathbf{w}_1, ..., \mathbf{w}_m \rangle$$

generated by the $\mathbf{v}_j, \mathbf{w}_j$. The reason is that each B_j tiles with a subgroup of Γ, the subgroup generated by its side vectors, and therefore it tiles with Γ multiply itself. Clearly, P is also a Nk-tiling of $\mathbb{R}^3$ by the union of N translations of Γ by the vectors $\mathbf{u}_1^+, ..., \mathbf{u}_N^+$.

Let $\{\gamma_j : j \in \mathbb{Z}\}$ be a complete set of representatives of G in Γ. Define the set of translates

$$\Lambda = \bigcup_{j\in\mathbb{Z}} (E_j + \gamma_j),$$

where for each $j \in \mathbb{Z}$ we choose $E_j = S$ or $E_j = T$ arbitrarily.
We claim that for any such choice of the E_j, $P + \Lambda$ is a Nk-tiling of $\mathbb{R}^3$. The claim is true if all $E_j = S$ as it is a restatement of the fact that P is a Nk-tiling of $\mathbb{R}^3$ with $\Gamma + \{\mathbf{u}_1^+ + ... + \mathbf{u}_N^+\}$. Observe now that if we change any single E_j from S to T we are adding the quantity

$$\mathbf{1}_P * \delta_G * \sum_{k=1}^{N} \delta_{\mathbf{u}_k^-} * \delta_{\gamma_j} - \mathbf{1}_P * \delta_G * \sum_{k=1}^{N} \delta_{\mathbf{u}_k^+} * \delta_{\gamma_j} \tag{4.12}$$

to the constant function

$$\mathbf{1}_P * \delta_\Lambda,$$

which therefore remains the same since (4.12) is identically zero by (4.11). We conclude that we have a Nk-tiling no matter how E_j the are chosen. Choose now all $E_j = S$ with the exception $E_0 = T$. We claim that the corresponding set Λ is not a finite union of translated 3-dimensional lattices. By the Poisson Summation Formula we have that, if

$$\Lambda' = \bigcup_{j\in\mathbb{Z}} (S + \gamma_j) = \Gamma + \{\mathbf{u}_1^+ + ... + \mathbf{u}_N^+\}$$

then $\hat{\delta}_{\Lambda'}$ is a discrete measure in $\mathbb{R}^3$ and this should also be true for $\hat{\delta}_\Lambda$, if Λ were a finite union of translated lattices. Thus the difference

$$\hat{\delta}_{\Lambda'} - \hat{\delta}_\Lambda$$

would also be a discrete measure. But

$$\begin{aligned}
\delta_{\Lambda'} - \delta_\Lambda &= \delta_{S+\gamma_0} - \delta_{S+\gamma_0} \\
&= \delta_{\gamma_0} * \delta_G * \sum_{k=1}^{N} (\delta_{\mathbf{u}_k^+} - \delta_{\mathbf{u}_k^-}) \\
&= \delta_{\gamma_0} * \delta_G * (\delta_0 - \delta_{c_1\mathbf{v}_1}) * ... * (\delta_0 - \delta_{c_n\mathbf{v}_n}).
\end{aligned}$$

Hence

$$\hat{\delta}_{\Lambda'} - \hat{\delta}_\Lambda = e^{2\pi i\langle\gamma_0,\xi\rangle} \prod_{j=1}^{n} (1 - e^{2\pi i c_j\langle\mathbf{v}_j,\xi\rangle}) \hat{\delta}_G. \tag{4.13}$$

But we recall that

$$\mathrm{supp}\hat{\hat{\delta}}_G = G^* + H_1^{\perp}.$$

Hence, the factors in the right hand side of (4.13) vanish at the set

$$\bigcup_{j=1}^{n} (\mathbb{Z}\frac{\mathbf{v}_j}{c_j} + \mathbf{v}_j^{\perp}). \tag{4.14}$$

Each set in this union consists of a series of planes normal to $\mathbf{v}_j$ and spaced by $(c_j|\mathbf{v}_j|)^{-1}$. Each of the straight lines that make up the support of $\hat{\delta}_G$ is parallel to each plane and, therefore, each such line is either entirely contained in (4.14) or is disjoint from it. It follows that, since the right hand side of (4.13) is not identically zero, its support contains at least one straight line, so it is not a discrete set, contradiction. $\square$

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