University of Crete

Department of Mathematics and Applied Mathematics

Master Thesis

Monotone quantities on complete Riemannian manifolds with non-negative Ricci curvature

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Contents

Abstract

In the current thesis we consider an n-dimensional non-compact and complete Riemannian manifold with $n \geq 3$. We then present three new monotonicity formulas which involve quantities that can be thought of as *generalized normalized area* and *volume* of balls in our manifold. Using these new monotonicity formulas, we derive a new gradient estimate for the Green function which improves a previous estimate by Cheng and Yau. The present work is based on the study of a recent paper by Tobias H. Colding [5], as well as the use of standard results of geometric analysis such as the *Bishop-Gromov* theorem and the *Bochner-Weitzenbock* formula.

Acknowledgements

Firstly, I would like to express my sincere gratitude to my supervisor, Professor Stathis Filippas, for his assistance, patience and support throughout the preparation of this study. His guidance helped me during my research and the writing of this thesis. I feel very honored collaborating with him.

Besides my advisor, I would like to thank the rest of my thesis committee Professor I. Platis and Professor A. Tertikas for their useful suggestions and comments. Also, I would like to thank especially Professor I. Platis for his support and his observations during all the past years of my undergraduate and graduate studies.

This thesis is dedicated to Menelia and to my family. Their support, encouragement and love, were the most significant factors that encouraged me to go on with my work.

Last but not the least, I would like to thank Christos and Alexandros for their important help, and for all the beautiful moments we spent together and the awesome discussions we had on the bench of the Natural History Museum.

Chapter 1

Preface

The current thesis aims at presenting the first parts of the paper of Tobias Holck Colding entitled: 'New monotonicity formulas for Ricci curvature and applications I' (2012) (see [5]). To this end concepts of geometrical analysis and comparison geometry are required.

The structure of the thesis is as follows. In Chapter II we first recall some basic concepts of the Riemannian geometry and then we present various Volume Comparison Theorems in the case one has bounds on the Ricci curvature of the manifold. The Chapter concludes with the proof of the standard Bishop-Gromov volume comparison theorem.

In Chapter III we introduce the Green function, which plays a major role in this work. We are interested in Riemannian manifolds which admit a positive Green's function, that is, non-parabolic manifolds. We review some criteria which guarantee non-parabolicity. We then give a detailed derivation of a technical but crucial identity that allows us to differentiate integral quantities where the integral is over a domain that is not fixed but depends on the variable of differentiation.

Chapters IV and V contain the main results of the work. In chapter IV we first define a "generalized distance" function by $b(y) = G^{\frac{1}{2-n}}(x, y)$, for fixed x. We then define the *normalized generalized area* by

$$A(r) = r^{1-n} \int_{b=r} |\nabla b|^3 d\text{Area} \ ,$$

and the *normalized generalized volume* by

$$V(r) = r^{-n} \int_{b \leq r} |\nabla b|^4 d\text{Vol} \ .$$

We then show the monotonicity formulas that involve A, V and b.

In chapter V we use the results of Chapter IV to obtain new global gradient estimates of the Green function that improve previously known estimates by Cheng-Yau, as well as a new asymptotic gradient estimate for the Green function near infinity.

Finally some useful formulas are collected in the Appendix.

Chapter 2

Geometrical Preliminaries and Volume Comparison Theorems

2.1 Rauch's Elementary Comparison Theorem

Comparison Geometry forms a central part of Riemannian Geometry. In this Chapter, follow Isaac Chavel's approach in [1], we quote some relevant theorems that compare volumes of balls and in general the geometry of a Riemannian manifold M to that of a simply connected model space M_δ of constant sectional curvature δ. Note that this model space for various values of δ, can take the following forms:

$$M_\delta = \begin{cases} \text{Sphere} & \text{if } \delta > 0 \\ \text{Euclidean space} & \text{if } \delta = 0 \\ \text{Hyperbolic space} & \text{if } \delta < 0 \ . \end{cases}$$

At the beginning, we introduce some notions which are important for this project:

Given a complete Riemannian manifold (M^n, g), we assume a unit speed geodesic $\gamma : [0, \beta] \to M$. A *Jacobi field* along γ is a differentiable vector field $Y(t)$ along γ, satisfying Jacobi's equation :

$$\nabla_t^2 Y + R(\gamma', Y)\gamma' = 0 \ .$$

We now define $\mathcal{J}$ to be the set of all Jacobi fields along γ, which is a vector space over $\mathbb{R}$ with dimension equal to 2n. Also $\mathcal{J}^\perp$ is the normal component of $\mathcal{J}$, which contains all non-zero Jacobi fields Y_i with

$$\langle Y_i, \gamma' \rangle = 0 \ ,$$

on all of $[0, \beta]$.

Moreover, we have:

Definition 2.1.1 *Given a real constant k, we let S_k denote the solution to the ordinary differential equation for $\psi(t)$,*

$$\psi'' + k\psi = 0$$

satisfying the initial conditions

$$S_k(0) = 0 \ , \qquad\qquad S_k'(0) = 1 \ .$$

We also let C_k denote the solution to the above ordinary differential equation satisfying the initial conditions

$$C_k(0) = 1 \ , \qquad\qquad C_k'(0) = 0 \ .$$

Solving the above differential equation with the corresponding initial conditions, we have for $S_k(t)$ that

$$S_k(t) = \begin{cases} (\frac{1}{\sqrt{k}}) \sin \sqrt{k}t & \textit{if } k > 0 \\ t & \textit{if } k = 0 \\ (\frac{1}{\sqrt{-k}}) \sinh \sqrt{-k}t & \textit{if } k < 0 \ . \end{cases}$$

Also for $C_k(t)$ we have,

$$C_k(t) = \begin{cases} \cos \sqrt{k}t & \textit{if } k > 0 \\ 1 & \textit{if } k = 0 \\ \cosh \sqrt{-k}t & \textit{if } k < 0 \ . \end{cases}$$

Furthermore, we have the following properties for these functions :

$$S_k'(t) = C_k(t), \qquad C_k'(t) = -kS_k(t), \qquad C_k^2(t) + kS_k^2(t) = 1$$

and

$$\left(\frac{C_k}{S_k}\right)'(t) = \left(\frac{S_k'}{S_k}\right)'(t) = -\frac{1}{S_k^2(t)} \ .$$

Also for two given linearly independent tangent vectors x and y, we define the *sectional curvature* $K(x, y)$ of the 2-section, determined by x, y as the

$$\mathcal{K}(x, y) := \frac{\langle R(x, y)x, y\rangle}{|x|^2|y|^2\langle x, y\rangle^2} \ ,$$

where R is the Riemann curvature tensor.

We now present our first Comparison Theorem.

Theorem 2.1.2 *(H.E. Rauch) Let M be a Riemannian manifold, δ a real constant, $\gamma : [0,\beta] \to M$ be a unit speed geodesic in M such that $K \leq \delta$ for all sectional curvatures along $\gamma \mid_{[0,\beta]}$. If $Y \in \mathcal{J}^{\perp}$, then the function $|Y|$, with $|Y| = \langle Y, Y \rangle^{\frac{1}{2}}$, along γ satisfies the differential inequality*

$$|Y|'' + \delta |Y| \geq 0 \ , \tag{2.1.1}$$

on $[0,\beta)$.

Furthermore, if $\psi(t)$ denotes the solution on $[0,\beta]$ of

$$\psi'' + \delta\psi = 0 \ , \qquad \psi(0) = |Y|(0) \ , \qquad \psi'(0) = |Y|'(0) \tag{2.1.2}$$

and $\psi(t)$ does not vanish on $(0,\beta)$, then on $(0,\beta)$ we have

$$\frac{d}{dt}\left(\frac{|Y|}{\psi}\right) = \left(\frac{|Y|}{\psi}\right)' \geq 0 \ , \qquad |Y| \geq \psi \ . \tag{2.1.3}$$

Equality holds in (2.1.3) at $t_0 \in (0,\beta)$ if and only if

$$\mathcal{K}(Y, \gamma') = \delta$$

on all of $[0, t_0]$, and in this case there exists a parallel vector field E along γ for which

$$Y(t) = \psi(t) E(t) \ ,$$

on all of $[0, t_0]$.

Proof. First we have that

$$\frac{d}{dt}|Y| = |Y|' = \langle Y, \nabla_t Y \rangle |Y|^{-1} \ ,$$

which implies

$$\frac{d^2}{dt^2}|Y| = |Y|'' = \{\langle \nabla_t Y, \nabla_t Y \rangle + \langle Y, \nabla_t^2 Y \rangle\}|Y|^{-1} - \langle Y, \nabla_t Y \rangle |Y|^{-2}|Y|' \ .$$

Since from Jacobi's equation $\nabla_t^2 Y = -R(\gamma', Y)\gamma'$, then we have:

$$\begin{aligned} |Y|'' &= \{|\nabla_t Y|^2 - \langle Y, R(\gamma', Y)\gamma' \rangle\}|Y|^{-1} - |Y|^{-3}\langle Y, \nabla_t Y \rangle^2 \\ &= |Y|^{-1}\langle Y, R(\gamma', Y)\gamma' \rangle + |Y|^{-3}|Y|^2|\nabla_t Y|^2 - |Y|^{-3}\langle Y, \nabla_t Y \rangle^2 \\ &\geq -\delta|Y| + |Y|^{-3}\{|Y|^2|\nabla_t Y|^2 - \langle Y, \nabla_t Y \rangle^2\} \\ &\geq -\delta Y \ , \end{aligned} \tag{2.1.4}$$

where for the last inequality we have used Cauchy-Schwarz inequality. This proves (2.1.1).

For the second claim, since

$$\left(\frac{|Y|}{\psi}\right)' = \frac{|Y|'\psi - |Y|\psi'}{\psi^2} ,$$

we set $F(t) := |Y|'\psi - |Y|\psi'$. We have for $F(t)$ that $F(0) = 0$ and

$$F'(t) = \{|Y|'\psi - |Y|\psi'\}'(t) \geq 0 ,$$

from (2.1.1), (2.1.2). These imply immediately that

$$F(t) = \{|Y|'\psi - |Y|\psi'\} \geq 0$$

and hence

$$(\frac{|Y|}{\psi})'(t) \geq 0 .$$

This, together with

$$\lim_{t\to 0} \frac{|Y|}{\psi} = 1 ,$$

implies that

$$|Y|(t) \geq \psi(t) \text{ on } (0, \beta) .$$

Now in the case of equality, if we have $(\frac{|Y|}{\psi})'(t) = 0$ at some $t_0 \in (0, \beta]$ then

$$\{|Y|'\psi - |Y|\psi\}'(t_0) = F(t_0) = 0 ,$$

which implies $F(t) = 0$ on all of $[0, t_0]$. and this, in turn, implies $|Y| = \psi$ on all of $[0, t_0]$. In this case, we may write $Y = \psi E$ with $|E| = 1$ and $\psi > 0$ along the geodesic γ . From the definition of the covariant derivative along a curve,

$$\nabla_t Y = \nabla_t(\psi E) = \psi'(t)E + \psi\nabla_t E . \tag{2.1.5}$$

Therefore we have equality in (2.1.1) on $(0, t_0]$, since

$$|Y|'' + \delta|Y| = \psi'' + \delta\psi = 0 .$$

Then we have equality in (2.1.4) and thus also in the Cauchy-Schwarz inequality, which implies $|\langle Y, \nabla_t Y\rangle| = |Y||\nabla_t Y|$. This holds if and only if Y and $\nabla_t Y$ are linearly dependent on all of $(0, t_0]$.
Now, suppose that $\nabla_t Y = \lambda Y$, with $\lambda \in \mathbb{R}$. Then by (2.1.5) we have

$$\psi'(t)E - \lambda\psi E = -\psi\nabla_t E ,$$

which implies

$$[\psi'(t) - \lambda\psi]E = \psi\nabla_t E .$$

But E and $\nabla_t E$ are linearly independent whenever $\nabla_t E$ is non-zero, since the Levi-Civita connection is compatible with the metric. Finally, since ψ is non-zero on all of $(0, t_0]$, we have that $\nabla_t E = 0$ on $(0, t_0]$, which in turn induces that E is parallel along γ on all of $[0, t_0]$.

□

2.2 Some Volume Comparison Theorems

In this section we will prove some comparison theorems under the assumption that the curvature is bounded either above or below. In the former case we will assume bounds on the Ricci curvature, while in the latter we need bounds for the sectional curvature. We will first restate Rauch's Comparison Theorem.

We need the following definitions :

i) $\mathbf{S}_p = \{\xi \in T_pM : |\xi| = 1\}$ and
ii) the matrix $A(t, \xi)$ is the solution of the matrix ordinary differential equation on $\xi^\perp : A'' + RA = 0$ satisfying the initial conditions $A(0, \xi) = 0$ and $A'(0, \xi) = I$.

Let $p \in M, \xi \in \mathbf{S}_p$ and we assume that for all sectional curvatures along γ_ξ there holds $\mathcal{K} \leq \delta$. Then for any Jacobi field Y along γ_ξ , pointwise orthogonal to γ_ξ and $Y|_{\gamma_\xi(0)} = Y|_p = 0$, we have in the spirit of Rauch's Theorem

$$\frac{|Y|'}{|Y|} \geq \frac{S'_\delta}{S_\delta} \ , \tag{2.2.1}$$

$$|Y| \geq |\nabla_t Y|(0) S_\delta \tag{2.2.2}$$

for all $t < \frac{\pi}{\sqrt{\delta}}$, where $\frac{\pi}{\sqrt{\delta}} := +\infty$ when $\delta \leq 0$.

Furthermore, we have equality in (2.2.1) at $t = t_0 \in (0, \frac{\pi}{\sqrt{\delta}}]$ if and only if there exists a parallel vector field E along γ_ξ such that

$$Y(t) = S_\delta E(t) \quad \textit{and} \quad R(t)E(t) = \delta E(t) \ , \tag{2.2.3}$$

for all $t \in [0, t_0]$.

(The proof is practically the same with the proof above, both for equality and inequality of (2.2.1).)

In particular we have

$$(A^*A)(t, \xi) \geq S_\delta^2(t) I \ , \tag{2.2.4}$$

where A^* denotes the adjoint of the linear transformation A, for all $t \in (0, \frac{\pi}{\sqrt{\delta}}]$.

Equality (2.2.4) holds because we have for A, that: $(A^*A)^* = A^*A^{**} = A^*A$ is self-adjoint and then

$$\begin{aligned} (A^*A)(t, \xi) \geq S_\delta^2(t) I &\Leftrightarrow \langle A^*Ax, x\rangle \geq \langle S_\delta^2 x, x\rangle \\ &\Leftrightarrow \langle Ax, Ax\rangle \geq \langle S_\delta x, S_\delta x\rangle \\ &\Leftrightarrow |Y| \geq S_\delta |Y|'(0) \ , \end{aligned}$$

which is valid on $(0, \frac{\pi}{\sqrt{\delta}}]$ by Rauch's theorem.

The equality in (2.2.4) at a $t_0 \in (0, \frac{\pi}{\sqrt{\delta}}]$ holds if and only if

$$A(t, \xi) = S_\delta(t) I \ , \qquad\qquad R(t) = \delta I \ , \tag{2.2.5}$$

for all $t \in (0, t_0]$.

This holds, because

$$(A^*A)(t,\xi) = S_\delta^2(t)I$$

is true if and only if equality holds in (2.2.2). Also, equality in (2.2.2) implies (2.2.3). Finally, since A is the matrix that gives us the solutions of the Jacobi equation along γ_ξ, (2.2.3) is valid on all of $(0, t_0]$ if and only if (2.2.5) is valid, on all of $(0, t_0]$.

Theorem 2.2.1 *(P.Gunther, R.L.Bishop) We assume we have the geodesic γ_ξ as described above, with all sectional curvatures along γ_ξ less than or equal to δ . Then*

$$\frac{(\det A)'}{\det A} \geq (n-1)\frac{S_\delta'}{S_\delta} \ , \tag{2.2.6}$$

on $(0, \frac{\pi}{\sqrt{\delta}})$ and

$$\det A \geq S_\delta^{n-1} \ , \tag{2.2.7}$$

on $(0, \frac{\pi}{\sqrt{\delta}}]$. We have equality in (2.2.6) at a $t_0 \in (0, \frac{\pi}{\sqrt{\delta}}]$, if and only if (2.2.5) is valid at each point of $[0, t_0]$.

Proof. We set the matrix $B := A^*A$ which is self-adjoint. Since $\det A = \det A^*$, we have easily that

$$\frac{(\det B)'}{(\det B)} = \frac{[(\det A)^2]'}{(\det A)^2} = \frac{2(\det A)'}{(\det A)} \Rightarrow \frac{(\det A)'}{(\det A)} = \frac{1}{2}\frac{(\det B)'}{(\det B)} \ .$$

We now consider $\tau \in (0, \frac{\pi}{\sqrt{\delta}})$ and an orthonormal basis $\{e_1, ..., e_{n-1}\}$ of $\xi^\perp$ consisting of eigenvectors of $B(\tau)$. Moreover we consider the solutions $\{\eta_1(t), ..., \eta_{n-1}(t)\}$ to the vector Jacobi equation in $\xi^\perp$:

$$\eta'' + R(t)\eta = 0 \ .$$

Observe that

$$\langle \eta_a(t), \eta_\beta(t)\rangle = \langle A(t)e_a, A(t)e_\beta\rangle = \langle A^*(t)A(t)e_a, e_\beta\rangle = \langle B(t)e_a, e_\beta\rangle = B_{a\beta}(t) \ .$$

Moreover for $t = \tau$ and $a \neq \beta \Rightarrow B_{a\beta}(\tau) = 0$, since

$$B_{a\beta}(\tau) = \delta_{a\beta}\langle \eta_a(\tau), \eta_\beta(\tau)\rangle \ .$$

Then, since for $(A_{jk}) \neq 0$ one has

$$\frac{\partial}{\partial x^l}[\ln(detA)] = \operatorname{tr}[\frac{\partial}{\partial x^l}(A)A^{-1}] \ ,$$

we have that

$$\frac{(\det A)'}{(\det A)}(\tau) = \frac{1}{2}\frac{(\det B)'}{(\det B)}(\tau) = \frac{1}{2}\operatorname{tr}(B'(\tau)B^{-1}(\tau))$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{a,\beta=1}^{n-1} B'_{a\beta}(\tau)B^{-1}_{\beta a}(\tau) \\
&= \frac{1}{2}\sum_{a,\beta=1}^{n-1} \{(\langle \eta'_a, \eta_\beta\rangle + \langle \eta_a, \eta'_\beta\rangle)\frac{\delta_{a\beta}}{\langle \eta_a, \eta_a\rangle}\}\big|_\tau \\
&= \sum_{a=1}^{n-1} \frac{\langle \eta'_a, \eta_a\rangle}{\langle \eta_a, \eta_a\rangle}(\tau) = \sum_{a=1}^{n-1} \frac{1}{2}\frac{(|\eta_a|^2)'}{|\eta_a|^2}(\tau) \\
&= \sum_{a=1}^{n-1} \frac{|\eta_a|'}{|\eta_a|}(\tau) = \sum_{a=1}^{n-1} \frac{|Y|'}{Y}\big|_a \\
&\geq (n-1)\frac{S'_\delta}{S_\delta}\ ,
\end{aligned}
$$

with the last inequality following from (2.2.1). Now, integrating (2.2.6) we have

$$
\int_0^t \left[\ln\left(\frac{\det A}{S_\delta^{n-1}}\right)\right]' dt \geq 0
$$

and since

$$
\lim_{t\to 0} \frac{\det A}{S_\delta^{n-1}} = 1\ ,
$$

get

$$
\det A \geq S_\delta^{n-1}\ ,
$$

on $(0, \frac{\pi}{\sqrt{\delta}}]$, which is (2.2.5).

The case of equality is immediate. If we have equality in (2.2.5) on $(0, \frac{\pi}{\sqrt{\delta}}]$, we have that

$$
\frac{|\eta_a|'}{|\eta_a|} = \frac{S'_\delta}{S_\delta}\ .
$$

Then from (2.2.3)

$$
A(t)e_a = \eta_a(t) = S_\delta(t)e_a\ , \qquad \text{for } a = 1, ..., n-1\ .
$$

Therefore we have equality in (2.2.1) and (2.2.2) which implies that (2.2.5) is valid at each point of $[0, t_0]$.

□

Before we state our next Theorem, we define some quantities on the manifold M as well as on its tangent bundle.

Given a Riemannian manifold M and γ a geodesic in M, a point $\gamma(t_1)$ is said to be *conjugate* to $\gamma(t_0)$ along γ if there exists $Y \in \mathcal{J}$, $Y \neq 0$ such that

$$
Y(t_0) = Y(t_1) = 0\ .
$$

The *cut point* of $p \in M$ along a curve γ_t is $\gamma(t_0)$, where t_0 is the supremum of the finite set $\{t : \gamma|_{[0,t]}$ is a minimizing geodesic$\}$. For every $p \in M$, the *tangential cut locus* of p in T_pM is

$$\mathbf{C}(p) := \{c(\xi)\xi : c(\xi) < +\infty, \xi \in S_p\}$$

and the *cut locus* of p in M

$$C(p) := \exp \mathbf{C}(p) \ ,$$

where $c(\xi)$ is the distance to the cut point of p along γ_ξ. Also, is defined

$$\mathbf{D}_p := \{t\xi : 0 \leq t < c(\xi), \xi \in S_p\}$$

and

$$D_p := \exp \mathbf{D}_p \ .$$

Finally, the *injectivity radius* of p, inj$p$, is the infimum of the set

$$\{c(\xi) : \xi \in S_p\} \ .$$

Note that the smallest distance from p to the cut locus of p is equal to $injp$.

Now we have $M \backslash C(p) = D(p)$ and thus, a chart on $D(p)$ is given by

$$(\exp_p |\mathbf{D}_p \setminus \{p\})^{-1} : D_p \backslash \{p\} \to \mathbf{D}_p \backslash \{p\} \ .$$

The Riemannian measure is given on D_p by

$$dV(\exp(t\xi)) = \sqrt{g(t,\xi)} dt d\mu_p(\xi)$$

for some function $\sqrt{g}$ on $\mathbf{D}_p$, where $d\mu_p(\xi)$ denotes the Riemannian measure on S_p induced by the Euclidean Lebesque measure on T_pM and

$$\sqrt{g(t,\xi)} = \det A(t,\xi) \ .$$

The next Theorem is about the volumes of a disk on M and M_δ.

Theorem 2.2.2 *(P.Gunther, R.L.Bishop) Assume that the sectional curvatures of M are all less than or equal to δ. Then for every $x \in M$ we have*

$$V(x,r) \geq V_\delta(r) \tag{2.2.8}$$

for all $r \leq \min\{\text{inj}x, \frac{\pi}{\sqrt{\delta}}\}$, with equality for some fixed r if and only if $B(x,r)$ is isometric to the disk of radius r in the constant curvature space form M_δ.

Proof. Fubini's Theorem together with the theorem 2.2.1 gives

$$\begin{aligned}
V(x,r) &= \int\limits_{\mathbf{D}_x} \sqrt{g}(t,\xi)dt d\mu_x(\xi) \\
&= \int\limits_{\mathbf{S}_x} d\mu_x(\xi) \int\limits_0^{\min\{c(\xi),r\}} \sqrt{g}(t,\xi)dt \\
&= \int\limits_{\mathbf{S}_x} d\mu_x(\xi) \int\limits_0^{r} \sqrt{g}(t,\xi)dt \\
&= \int\limits_{\mathbf{S}_x} d\mu_x(\xi) \int\limits_0^{r} \det A(t,\xi)dt \\
&\geq \int\limits_{\mathbf{S}_x} d\mu_x(\xi) \int\limits_0^{r} S_\delta^{n-1}(t)dt \\
&= V_\delta(r) \ .
\end{aligned}$$

The last equality holds since

$$\int\limits_{\mathbf{S}_x} d\mu_x(\xi) = \int\limits_{|\xi|=1} d\mu_x(\xi) = c_{n-1}$$

and a disk of radius r in M_δ has volume equal to $V_\delta(r) = c_{n-1}\int\limits_0^r S_\delta^{n-1}(t)dt$.

If we have equality in the above inequality, it follows that for every ξ, t

$$\det A(t,\xi) = S_\delta^{n-1}(t) \ ,$$

which implies that

$$A(t,\xi) = S_\delta I \ .$$

Therefore g equals the metric of constant curvature δ in polar coordinates and $B(x,r)$ is isometric to $V_\delta(r)$.

□

Now, as we said at the beginning of this Chapter, we have some similar volume comparison theorems in which the curvature is bounded from below, where the final of these is the Bishop-Gromov theorem, which is the goal of this chapter.

Theorem 2.2.3 *(R.L. Bishop) We assume we are given a fixed geodesic γ_ξ, with the Ricci curvature along γ_ξ greater than or equal to $(n-1)k$, that is,*

$$\mathrm{Ric}(\gamma_\xi'(t), \gamma_\xi'(t)) = \mathrm{trR}(t) \geq (n-1)k \ , \tag{2.2.9}$$

for all $t \in (0, c(\xi)]$*. Then*

$$\frac{(\det A)'}{\det A} \leq (n-1)\frac{S_k'}{S_k} \ , \tag{2.2.10}$$

in $(0, c(\xi))$*, and*

$$\det A \leq S_k^{n-1} \ , \tag{2.2.11}$$

in $(0, c(\xi)]$*.*

Equality holds in (2.2.10) *at* $t = t_o \in (0, c(\xi)]$ *if and only if*

$$A(t) = S_k(t) I \quad \textit{and} \quad R(t) = kI \ , \tag{2.2.12}$$

for all $t \in (0, t_0]$*.*

Remark: Note that (2.2.11) implies the non-existence of a conjugate point after the first zero of $S_k(t)$, i.e. after $\frac{\pi}{\sqrt{k}}$, unless $\frac{\pi}{\sqrt{k}}$ is the conjugate point. With other words, when $k > 0$ $\det A$ haven't a zero later than $\frac{\pi}{\sqrt{k}}$.

Proof. Set $Ct_k(t) := \frac{S_k'(t)}{S_k(t)}$ and consider $\psi(t) := (n-1)Ct_k$. Calculations show that the function ψ satisfies the scalar Riccati equation

$$\psi'(t) + \frac{\psi^2(t)}{(n-1)} + (n-1)k = 0 \ .$$

Also the function $\psi(t) = (n-1)\frac{S_k'(t)}{S_k(t)}$ satisfies:

• It is strictly decreasing with respect to t, for each t.
• When $k \leq 0$, has limiting value, as $t \to \infty$, equal to $(n-1)\sqrt{-k}$, because when $k < 0$,

$$\lim_{t\to\infty} \psi(t) = (n-1) \lim_{t\to\infty} \frac{S_k'(t)}{S_k(t)} = (n-1) \lim_{t\to\infty} \frac{[(\frac{1}{\sqrt{-k}})\sinh(\sqrt{-k}t)]'}{(\frac{1}{\sqrt{-k}})\sinh(\sqrt{-k}t)}$$
$$= (n-1)\sqrt{-k} \lim_{t\to\infty} \frac{\cosh(\sqrt{-k}t)}{\sinh(\sqrt{-k}t)} = (n-1)\sqrt{-k}$$

with $\lim_{t\to\infty} \frac{\cosh(\sqrt{-k}t)}{\sinh(\sqrt{-k}t)} = 1$ and when $k = 0$,

$$\lim_{t\to\infty} \psi(t) = (n-1) \lim_{t\to\infty} \frac{S_k'(t)}{S_k(t)} = (n-1) \lim_{t\to\infty} \frac{1}{t} = 0 \ .$$

Furthermore for linear transformations $K(t), L(t) : V \longrightarrow V$, where V is a finite dimensional vector space, their Wronskian $W(t)$ is defined by

$$W(K, L) := K'^* L - K^* L' \ .$$

From this we have that $W(A,A)=0$ because

$$\begin{aligned}\frac{d}{dt}W(A,A) &= \frac{d}{dt}[A'^*(t)A(t)-A^*(t)A'(t)] \\ &= [(A'^*)'A + A'^*A'](t) - [(A^*)'A' + A^*A''](t) \\ &= (A'^*)'(t)A(t) - A^*(t)A''(t) \ .\end{aligned}$$

Now since $(A')^* = (A^*)'$ and because of $A'' + R(t)A = 0$, we have

$$(A'^*)' = (A^*)'' = (A'')^* = (-R(t)A)^* = -A^*R^*(t) \ . \tag{2.2.13}$$

Of course, we have also for $R(t) = R(\cdot,\gamma'(t))\gamma'(t) : T_{\gamma(t)}M \longrightarrow T_{\gamma(t)}M$ that, if X,Y are differentiable vector fields on M :

$$\begin{aligned}R(t) = R^*(t) &\Leftrightarrow \langle R(t)X,Y\rangle = \langle X,R(t)Y\rangle \\ &\Leftrightarrow \langle R(X,\gamma')\gamma',Y\rangle = \langle R(Y,\gamma')\gamma',X\rangle \ .\end{aligned}$$

With this and (2.2.12) we conclude

$$\frac{d}{dt}W(A,A) = -A^*R(t)A + A^*R(t)A = 0 \ .$$

Moreover

$$W(A(t),A(t))(0) = W(A(0),A(0)) = (A'(0))^*A(0) - A^*(0)A'(0) = 0 \ ,$$

by the initial conditions of $A'' + R(t)A = 0$.

All above give us:

$$W(A,A) = 0 \ .$$

Now for the matrix $U := A'A^{-1}$, we can see that is self-adjoint. Indeed

$$\begin{aligned}U^* - U &= (A^{-1})^*A'^* - A'A^{-1} \\ &= (A^{-1})^*[A'^*A - A^*A']A^{-1} \\ &= (A^{-1})^*W(A,A)A^{-1} \\ &= 0 \ .\end{aligned}$$

One more property about the matrix U, is that it satisfies the matrix Riccati equation $U' + U^2 + R = 0$.

Now we have that

$$\mathrm{tr}(U') = \sum_i u'_{ii} = (\sum_i u_{ii})' = (\mathrm{tr}U)' \ ,$$

and then

$$(\mathrm{tr}U)' + \mathrm{tr}U^2 + \mathrm{tr}R = 0 \ . \tag{2.2.14}$$

Moreover the Cauchy-Schwarz inequality implies

$$(\mathrm{tr}U)^2 \le (n-1)\mathrm{tr}(U^2)\ . \tag{2.2.15}$$

To see this, suppose that $\lambda_1, ..., \lambda_{n-1}$ are the eigenvalues of U. Then

$$\begin{aligned}(\mathrm{tr}U)^2 = (\lambda_1 + ... + \lambda_{n-1})^2 &= \sum_{i=1}^{n-1} \lambda_i^2 + \sum_{i\neq j} 2\lambda_i\lambda_j \\ &\le \sum_{i=1}^{n-1} \lambda_i^2 + \sum_{i\neq j} (\lambda_i^2 + \lambda_j^2) \\ &= \sum_{i=1}^{n-1} \lambda_i^2 + (n-2)\sum_{i=1}^{n-1} \lambda_i^2 \\ &= (n-1)\sum_{i=1}^{n-1} \lambda_i^2 \\ &= (n-1)\mathrm{tr}(U^2)\ .\end{aligned}$$

For $\phi := \mathrm{tr}U = \mathrm{tr}A'A^{-1} = \frac{(\det A)'}{\det A}$, (2.2.15) implies that

$$\phi'(t) + \frac{\phi^2(t)}{n-1} + \mathrm{tr}R(t) \le 0\ ,$$

which implies, since $\mathrm{tr}R(t) \ge (n-1)k$ and the above inequality, the differential inequality

$$\phi'(t) + \frac{\phi^2(t)}{n-1} + (n-1)k \le 0\ . \tag{2.2.16}$$

Now we wish to compare ϕ with ψ. Let the function f be

$$f(t) = S_k''S_k - (S_k')^2 = -kS_k^2 - (S_k')^2\ .$$

Then

$$\begin{aligned}f'(t) &= -2kS_kS_k' - 2S_k'S_k'' \\ &= -2kS_kS_k' + 2kS_kS_k' = 0\ .\end{aligned}$$

Also,

$$\begin{aligned}f(0) &= [-kS_k^2 - (S_k')^2](0) \\ &= -(S_k')^2 < 0\ ,\end{aligned}$$

and from these follows that $\psi'(t) < 0$, and then

$$\Psi := \frac{\psi^2}{n-1} + (n-1)k > 0\ ,$$

on all of $(0, \frac{\pi}{\sqrt{k}})$.

Taking the Taylor series for A, we have

$$A = tI + o(t) \ .$$

From this, it follows that

$$A^{-1}(t) = t^{-1}I + o(t^{-1}) \ , \tag{2.2.17}$$

and as before

$$A'(t) = I + o(t) \ . \tag{2.2.18}$$

From (2.2.17) and (2.2.18) we have

$$\begin{aligned} \phi = \text{tr}A'A^{-1} &= \text{tr}(t^{-1}I) + \text{tr}(o(t^{-1})) \\ &= (n-1)t^{-1} + o(t^{-1}) \ . \end{aligned}$$

Therefore, $\phi \sim \frac{n-1}{t}$, as $t \downarrow 0$.

So, there exists an $\varepsilon_0 > 0$ such that

$$\Phi := \frac{\phi^2}{n-1} + (n-1)k > 0 \ ,$$

on $(0, \varepsilon_0)$. Assume that $\Phi > 0$ on all of $(0, t)$, $t \in (0, \text{conj}\xi)$. Then the differential inequality (2.2.16) implies

$$\frac{-\phi'}{\frac{\phi^2}{n-1} + (n-1)k} \geq 1 \ , \tag{2.2.19}$$

which impies

$$\int_0^s \frac{-\phi'}{\frac{\phi^2}{n-1} + (n-1)k}(\tau)d\tau \geq s, \forall s \in (0, t] \ . \tag{2.2.20}$$

From the last inequality we have $\phi \leq \psi$ on $(0, t]$, because

$$\int_0^s \frac{-\phi'}{\frac{\phi^2}{n-1} + (n-1)k}(\tau)d\tau = \text{arc}Ct_k\left(\frac{\phi(s)}{n-1}\right) \ , \tag{2.2.21}$$

where $\text{arc}Ct_k$ is the inverse function of Ct_k. To prove (2.2.21) it is enough to show that

$$\text{arc}Ct_k\left[\frac{\phi(s)}{n-1}\right]' = \frac{-\phi'}{\frac{\phi^2}{n-1} + (n-1)k}$$

and since $\phi(0) = \infty$ from $\phi \sim \frac{n-1}{t}$, we have $Ct_k(0) = \infty$. Then integrating from 0 to s we have (2.2.21). Then we have by letting $\frac{\phi(s)}{n-1}$, that

$$\left[\text{arc}Ct_k\left(\frac{\phi(s)}{n-1}\right)\right]' = \frac{d}{dx}[\text{arc}Ct_k(x)]\ \frac{d}{ds}\left[\frac{\phi(s)}{n-1}\right]$$

$$= \frac{1}{[Ct_k(x)]'}\Big|_{x=Ct_k{}^{-1}\left(\frac{\phi(s)}{n-1}\right)} \frac{\phi'(s)}{n-1} \ . \tag{2.2.22}$$

Now from the properties of functions $S_k(t)$, $C_k(t)$ and the definition of Ct_k, we have

$$\begin{aligned}(Ct_k)' &= \frac{S''(t)}{S(t)} - \left[\frac{S'(t)}{S(t)}\right]^2 \\ &= \frac{S''(t)}{S(t)} - Ct_k^2 \\ &= -k - Ct_k^2 \ .\end{aligned}$$

With the last computation, (2.2.22) becomes

$$\begin{aligned}\left[\mathrm{arc}Ct_k\left(\frac{\phi(s)}{n-1}\right)\right]' &= -\frac{1}{k + \frac{\phi^2(t)}{(n-1)^2}} \frac{\phi'(s)}{n-1} \\ &= \frac{-\phi'}{\frac{\phi^2}{n-1} + (n-1)k} \ ,\end{aligned}$$

which is the claim. Now, (2.2.11) follows from (2.2.10).

Equality in (2.2.10) at some $t_0 > 0$, means that we have $\phi(t_0) = \psi(t_0)$ and since these functions both satisfy the Riccati equation, at this t_0 we have that

$$\frac{-\phi'}{\frac{\phi^2}{n-1} + (n-1)k} = 1$$

and hence equality in (2.2) too. This implies equality in (2.2.16) on all of $(0, t_0]$ and then the equality in the Cauchy-Scharz inequality above, since

$$\begin{aligned}0 &= (\mathrm{tr}U)' + \mathrm{tr}(U^2) + \mathrm{tr}R \\ &\geq \mathrm{tr}U' + \frac{(\mathrm{tr}U)^2}{n-1} + (n-1)k \\ &= \phi' + \frac{\phi^2}{n-1} + (n-1)k = 0 \ .\end{aligned}$$

Moreover, the above impies

$$trR = (n-1)k \ ,$$

on all of $(0, t_0]$. Now, since U is self-adjoint and satisfies

$$\mathrm{tr}(U^2) = \frac{(\mathrm{tr}U)^2}{n-1} \ ,$$

which is a property only of multiples of the identity matrices, we have that

$$U = \lambda I$$

with $\lambda \in \mathbb{R}$ and for all $t \in (0, t_0]$. Then from matrix Riccati equation follows that R is scalar multiple of the identity for each t, because

$$U' + U^2 + R = 0$$

and from this it follows

$$(\lambda I)' + (\lambda I)^2 + R = 0$$

and then

$$R = -\lambda^2 I = \lambda' I \ .$$

Also, since

$$\mathrm{tr}U = \mathrm{tr}(\lambda I) = (n-1)\lambda$$

and

$$\mathrm{tr}U = \mathrm{tr}(A'A^{-1})(t) = \frac{(\det A)'}{\det A} = (n-1)\frac{S_k'}{S_k}(t) \ \forall t \in (0, t_0] \quad ,$$

we have that $\lambda = \frac{S_k'}{S_k}$ and then $R(t) = kI$ and $A(t) = S_k(t)I$ for all $t \in (0, t_0]$, which is (2.2.12).

It remains to consider the case of a given point $t \in (0, \mathrm{conj}\xi]$, for which the inequality $\Phi \ > 0$ is not valid on all of $(0, t]$ and assume that we do not have $\phi(t) \le \psi(t)$ on all of $(0, t)$. Let $S := \{t > 0, \phi(s) \le \psi(s), \forall s\}$ and the assumption that $\phi > \psi$. Then since the $\sup(S) := t_1$ exists, we have $\phi \le \psi$ on $(0, t_1)$ and $\phi(t_1) = \psi(t_1)$. Then $\Phi(t_1) \ > \ 0$ and since $\Phi(t)$ is continuous, there exists $\varepsilon_1 \ > \ 0$ such that $\Phi|_{[t_1, t_1+\varepsilon_1)} \ > \ 0$. This implies that (2.2.19) is valid on all of $[t, s)$, with $s \in (t_1, t_1 + \varepsilon_1)$ and from this follows $\phi \le \psi$ on $(t_1, t_1 + \varepsilon_1)$, which is a contradiction to the fact that $t_1 = \sup(S)$. Then we have (2.2.10) on all of $(0, t]$ and this holds also in the case of equality, by similar arguments we may prove.

□

2.3 Bishop-Gromov's Volume Comparison Theorem

From now on, we will write the volume of a geodesic ball in M with radius r centered at some $x \in M$, as $\mathrm{Vol}(B_x(r))$, instead of $V(x, r)$ and $\mathrm{Vol}(B_x^k(r))$, instead of $V_k(r)$ respectively.

Considering this lower bound for the Ricci curvature, we have the Bishop's Theorem in which in contrast to Gunter-Bishop Theorem, the opposite inequality is holding in the relationship of the volumes we are interested in. With this theorem we are able to compare volumes of balls in our manifold with the volumes of the respective balls in manifolds with constant sectional curvature.

Theorem 2.3.1 *(R.L.Bishop) Assume that the sectional curvatures of M are all greater than to $(n-1)k$. Then for every $x \in M$ and every $r > 0$, we have*

$$\mathrm{Vol}(B_x(r)) \leq \mathrm{Vol}(B_x^k(r)) \ , \tag{2.3.1}$$

for all $r \leq \min\{\mathrm{inj}x, \frac{\pi}{\sqrt{\delta}}\}$, with equality for some fixed r if and only if $B(x,r)$ is isometric to the disk of radius r in the constant curvature space form M_δ.

Proof. The proof is almost the same with the proof of Gunther-Bishop's theorem before.

□

A technical result follows, which we will need in the proof of the main theorem of this Chapter.

Lemma 2.3.2 *(M. Gromov) Suppose that f and g are positive integrable functions, of a real variable r, for which*

$$f/g$$

is decreasing with respect to r. Then the function

$$\int_0^r f / \int_0^r g$$

is also decreasing with respect to r.

Proof. For $r < R$ we have that

$$\int_0^r f \int_0^R g = \int_0^r f \int_0^r g + \int_0^r f \int_r^R g \tag{2.3.2}$$

and

$$\int_0^R f \int_0^r g = \int_0^r f \int_0^r g + \int_r^R f \int_0^r g \ . \tag{2.3.3}$$

Set $f = gh$ and by hypothesis h is decreasing with respect to r. This implies that

$$\int_0^r f \int_r^R g = \int_0^r gh \int_r^R g \geq \int_0^r g \ h(r) \int_r^R g = h(r) \int_0^r g \int_r^R g \ .$$

Moreover we have

$$h(r) \int_0^r g \int_r^R g = \int_0^r g \int_r^R g \ h(r) \geq \int_0^r g \int_r^R gh = \int_0^r g \int_r^R f.$$

Thus we have

$$\int_0^r f \int_r^R g \geq \int_0^r g \int_r^R f \ . \tag{2.3.4}$$

But from (2.3.2), (2.3.3) and (2.3.4)

$$\int_0^r f \int_0^R g - \int_0^r f \int_0^r g \geq \int_0^R f \int_0^r g - \int_0^r f \int_0^r g$$

and hence,

$$\int_0^r f \int_0^R g \geq \int_0^R f \int_0^r g$$

which means that the function $\int_0^r f / \int_0^r g$ is decreasing with respect to r.

□

Theorem 2.3.3 *(Bishop, Gromov) Assume that the Ricci curvatures of M are all greater than or equal to $(n-1)k$. Then for every $x \in M$, we have*

$$\frac{\mathrm{Vol}(B_x(r))}{\mathrm{Vol}(B_x^k(r))} \ , \tag{2.3.5}$$

is decreasing with respect to r.

Proof. Consider $r < R$. First note that $\mathbf{D}_x(R) \subseteq \mathbf{D}_x(r)$. We have that the function

$$F(r) = \frac{\int\limits_{\mathbf{D}_x(r)} \det A(r,\xi) d\mu_x(\xi)}{c_{n-1} S_k^{n-1}(r)}$$

is decreasing with respect to r, because

$$\begin{aligned} F(r) &= \frac{\int\limits_{\mathbf{D}_x(r)} \det A(r,\xi) d\mu_x(\xi)}{c_{n-1} S_k^{n-1}(r)} \\ &= \frac{1}{c_{n-1}} \int\limits_{\mathbf{D}_x(r)} \frac{\det A(r,\xi)}{S_k^{n-1}(r)} d\mu_x(\xi) \\ &\geq \frac{1}{c_{n-1}} \int\limits_{\mathbf{D}_x(R)} \frac{\det A(r,\xi)}{S_k^{n-1}(r)} d\mu_x(\xi) && (2.3.6) \\ &\geq \frac{1}{c_{n-1}} \int\limits_{\mathbf{D}_x(R)} \frac{\det A(R,\xi)}{S_k^{n-1}(R)} d\mu_x(\xi) && (2.3.7) \end{aligned}$$

$$= \frac{\int\limits_{\mathbf{D}_x(R)} \det A(R,\xi) d\mu_x(\xi)}{c_{n-1} S_k^{n-1}(R)} \ .$$

Note that (2.3.6) and (2.3.7) hold since $\mathbf{D}_x(R) \subseteq \mathbf{D}_x(r)$ and $\frac{\det A(t,\xi)}{S_k^{n-1}(t)}$ is decreasing with respect to r from Bishop-Gromov, respectively.

Now the claim follows since, together with the above Lemma, it holds for the volume of a geodesic ball of radius r in M and also for a disk of radius r in M_δ, that

$$\text{Vol}(B_x(r)) = \int\limits_0^r \int\limits_{\mathbf{D}_x(r)} \det A(r,\xi) d\mu_x(\xi)$$

and

$$\text{Vol}(B_x^k(r)) = \int\limits_0^r c_{n-1} A_k(t) dt \ ,$$

respectively.

□

A direct conclusion of the Bishop-Gromov theorem, is the following Corrolary, which is proved by Peter Li (see [11], p. 16-17, Theorem 2.5) and is based on a paper written by Yau (see [21]).

Corollary 2.3.4 *Let M be an n-dimensional complete and non-compact Riemannian manifold, with nonnegative Ricci curvature.Then M has infinite volume.*

Proof. Let p be an arbitary point in M and $x \in \partial B_p(1+r)$, with $B_p(1+r)$ be a geodesic ball in M. By the Bishop-Gromov theorem for the geodesic balls with radius $2+r =: s$ and $r =: t$, both centered at x, we have

$$\frac{\text{Vol}(B_x(s))}{\text{Vol}(B_x^k(s))} \leq \frac{\text{Vol}(B_x(t))}{\text{Vol}(B_x^k(t))} \ .$$

For $k = 0$ and hence $M_0^n = \mathbb{R}^n$, we get

$$\text{Vol}(B_x(s)) \leq \text{Vol}(B_x(t)) \frac{s^n \text{Vol}(B_0(1))}{t^n \text{Vol}(B_0(1))}$$

$$\Rightarrow \text{Vol}(B_x(s)) - \text{Vol}(B_x(t)) \leq \text{Vol}(B_x(t)) \frac{s^n - t^n}{t^n} \ . \tag{2.3.8}$$

Now, from the fact that the distance between p and x is $d(p,x) = 1+r$, we have

$$B_p(1) \subset (B_x(2+r) \setminus B_x(r))$$

and therefore the same holds for their volumes, that is,

$$\mathrm{Vol}(B_p(1)) \le \mathrm{Vol}(B_x(2+r)) - \mathrm{Vol}(B_x(r)) \ . \tag{2.3.9}$$

Also, since

$$B_x(r) \subset B_p(1+r) \ ,$$

we have

$$\mathrm{Vol}(B_x(r)) \le \mathrm{Vol}(B_p(1+r)) \ .$$

Combining, (2.3.8) and (2.3.9) we conclude that

$$\begin{aligned} \mathrm{Vol}(B_p(1)) &\le \mathrm{Vol}(B_p(1+2r))\frac{s^n - t^n}{t^n} \\ &= \mathrm{Vol}(B_p(1+2r))\frac{(2+r)^n - r^n}{r^n} \ . \end{aligned} \tag{2.3.10}$$

Note that $\frac{(2+r)^n - r^n}{r^n} = \frac{C}{r}$, for $r \to \infty$ and C be a positive constant depending only on n.

Finally, from (2.3.8), (2.3.9) and (2.3.10) we have that the volume of the geodesic ball increases like r

$$\mathrm{Vol}(B_p(1)) \le \mathrm{Vol}(B_p(1+2r))\frac{(2+r)^n - r^n}{r^n} \ ,$$

and for $r \to \infty$,

$$\mathrm{Vol}(B_p(r)) \ge \frac{r}{C(n)}\mathrm{Vol}(B_p(1)) = \infty \ .$$

□

Chapter 3

The Green function on Complete Riemannian Manifolds

In this thesis, our general interest is how the curvature of a manifold affects the solution of a differential equation and in particular the Laplace equation. Thus from this point, an important role will play the Green's function which will be used for the defining of some quantities and the behavior of them will give us some estimates for the Green's function itself (Chapters IV, V).

3.1 Existence of Green's function and Parabolic Manifolds

In this section we recall some well-known facts about the existence of Green's function on manifolds and its basic properties. This is a topic studied by several mathematicians (e.g. Cheng, Yau, Li, Varopoulos, etc.); in the sequel we will quote their results.

First of all, we have in general for the Green's function on manifolds, that:

Let (M^n, g) be a Riemannian manifold, with or without boundary. Then a function $G : M \times M \setminus \{\{x, x\} : x \in M\} \to \mathbb{R}$ is called a Green's function on M, if it satisfies :

• $G(x, y) = G(y, x)$ i.e. is symmetric,
• for each y in M we have that the function $G_x(y) \in C^2(M \setminus \{x\})$ (as a function of x),
• $\Delta_y G(x, y) = -\delta_x(y)$ for all $x \neq y$,

where Δ is the Laplace-Beltrami[1]orerator and $\delta_x(y)$ is the Delta Dirac function.

[1]The Laplace-Beltrami operator, like the Laplacian, is defined to be the divergence of the gradient. You can also see how it is written in terms of a local coordinate system: Appendix 6.1, Bochner-Weitzenbock formula.

In the rest of this thesis, we will use for the Green function, the normalization (see also [7])

$$\Delta G(x, \cdot) = (2-n)\mathrm{Vol}(\mathbb{S}^{n-1})\delta_x \ ,$$

with $n \geq 3$. $\mathbb{S}^{n-1}$ denotes the standard $(n-1)$-dimensional sphere of radius 1. With this normalization we have

$$G(x, y) = |x-y|^{2-n}$$

in $\mathbb{R}^n$.

Now, we will show the distinguish between two cases; that is when a Riemannian manifold is compact and non-compact.

If we have a compact Riemannian manifold (M^n, g) with boundary, the Green's function, let $G(x, y)$, exists and is unique. T. Aubin and Tsutomu Hirosima constructed Green's function. In this case, Hirosima to construct $G(x, y)$ (see [17]) used harmonic coordinates[2] and found some estimates for G near the singularity.

Now in the case that we have a non-compact and complete Riemannian manifold (M^n, g), without boundary (see [19]), Bernard Malgrange ([22]) proved in 1955 the existence of a symmetric Green's function. Also, in 1987 Li-Tam ([12]) gave an alternative constructive argument for the existence of this symmetric Green's function.

At this point, we have the following definition (see [10]):

Definition 3.1.1 *A complete Riemannian manifold is said to be non-parabolic if it admits a positive Green function. Otherwise, is said to be parabolic.*

After that, the question when a complete Riemannian manifold is or isn't parabolic involved many mathematicians.

The parabolicity or not, of a manifold, is associated with the existence of a positive Green's function. More precisely (see [23]), we have that if $G(x, y) =: G_x(y)$ is a Green's function with pole at some $x \in \Omega \subset M$ and Dirichlet boundary condition on $\partial\Omega$, i.e.

$$\begin{cases} \Delta G = \delta_x & \text{in } \Omega \\ G = 0 & \text{on } \partial\Omega \ , \end{cases}$$

then either there exists a positive Green's function $G_x(y)$ (which is called minimal positive Green function), or there does not exist.

We confine ourselves in the case which exists a positive Green's function and then we choose the minimal one to be our Green's function. Note that if there

[2] On a Riemannian manifold (M^n, g), harmonic coordinates are a coordinate system $\{x^1, ..., x^n\}$ each of whose coordinate functions x^i is harmonic, meaning that it satisfies Laplace's equation $\Delta x^i = 0$.

exists a positive Green's function, there exists also the minimal one. This becomes true by looking its value at infinity: if it has limit which equals to zero , then this is the minimal positive function that we are interested in. If not, we subtract the liminf of its and thus we get the minimal positive Green's function.

In the case when (M^n, g) is parabolic, let x a fixed point in M and $B_x(r)$ is the geodesic ball of radius r, centered at x.

In 1975 Cheng and Yau, in [4], showed something relevant in this direction, i.e., one has for some $x \in M$ that if we have

$$\liminf_{r \to +\infty} \frac{\mathrm{Vol}_g(B_x(r))}{r^2} < +\infty \ ,$$

then (M^n, g) is parabolic.

Later, Grigoryan in [16] and Varopoulos in [20], proved that if for some $x \in M$ we have

$$\int_1^\infty \frac{r}{\mathrm{Vol}_g(B_x(r))} dr = +\infty \ ,$$

then (M^n, g) is again parabolic. Moreover, Varopoulos proved that if we have non-negative Ricci curvature and also

$$\int_1^\infty \frac{r}{\mathrm{Vol}_g(B_x(r))} dr < +\infty \ ,$$

then (M^n, g) is non-parabolic.

Applying the above results we have, knowing that the volume of an n-dimensional ball of radius r in $\mathbb{R}^n$

$$V_n(r) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} r^n \ ,$$

that for $n = 2$, since $\mathrm{Vol}(B_x(r)) = \pi r^2$ and

$$\lim_{b \to +\infty} \int_1^b \frac{1}{\pi r} dr = \infty \ ,$$

$\mathbb{R}^2$ is parabolic. More generally, as a direct consequence of Bishop-Gromov theorem, any 2-dimensional surface with Ric≥ 0, is parabolic.

Also, since the volume of a ball in $\mathbb{R}^3$ is $\mathrm{Vol}(B_x(r)) = \frac{4}{3}\pi r^3$, we have by a simply computation that

$$\frac{3}{4\pi} \lim_{b \to +\infty} \int_1^b \frac{1}{r^2} dr < \infty \ ,$$

and hence $\mathbb{R}^3$ is non-parabolic[3].

As we conclude, the parabolicity of a manifold is characterized by the behavior of the volumes of the geodesic balls, asymptotically at infinity. If these volumes are upper than r^2, then we say that the manifold is non-parabolic and if they are until r^2, the manifold is said to be parabolic.

3.2 Green fuction and some useful results

In this Chapter, we mention an intermediate result of a paper written by Lei-Ni([13]).

Let $\Omega \subseteq M$ be a bounded domain in M and $x \in \Omega$. Denote by $G_\Omega(x, y)$ the Green's function with Dirichlet boundary condition on $\partial\Omega$.

By the maximum principle for harmonic functions, we have $G_\Omega(x, y) > 0$ for any $x, y \,\epsilon\, \Omega$. Note that, $G_\Omega(x, y)$ is a harmonic function with pole at x.

Now we may define the G_Ω-*sphere*

$$\Psi_r := \{y | G_\Omega(x, y) = r^{-a}\}$$

with $a > 0$ which is by Sard's theorem, a smooth hypersurface in Ω for almost every r.
To continue, if we let $\phi_r(y) = G_\Omega(x, y) - r^{-a}$, we have respectively the G_Ω-*ball*

$$\Omega_r = \{y | \phi_r(y) > 0\}.$$

Proposition 3.2.1 *Let $v : \Omega \subseteq M \to R^n$ be a smooth function. Then for every $r > 0$,*

$$v(x) = \frac{1}{r^a} \int\limits_{\Omega_r} |\nabla \log G_\Omega|^2 v d\mu - \frac{a}{r^a} \int\limits_0^r \eta^a \int\limits_{\Omega_\eta} \phi_\eta \Delta v d\mu \frac{d\eta}{\eta} \;. \tag{3.2.1}$$

Proof. We first show that for almost every $r > 0$, $v(x)$ can be also written as

$$v(x) = \int\limits_{\Psi_r} |\nabla G_\Omega| v dA_y - \int\limits_{\Omega_r} \phi_r \Delta v d\mu_y \;. \tag{3.2.2}$$

By Green's second identity on a Riemannian manifold

$$\int\limits_{\Omega_r} [(\Delta G_\Omega)v - G_\Omega(\Delta v)] d\mu = \int\limits_{\Psi_r} (\frac{\partial G_\Omega}{\partial \nu} v - \frac{\partial v}{\partial \nu} G_\Omega) dA$$

and by the fact that $\Delta_y G_\Omega(x, y) = -\delta_x(y)$, we have

$$v(x) = -\int\limits_{\Psi_r} (\frac{\partial G_\Omega}{\partial \nu} v - \frac{\partial v}{\partial \nu} G_\Omega) dA - \int\limits_{\Omega_r} G_\Omega(\Delta v) d\mu \;. \tag{3.2.3}$$

[3]In general, is true that $\mathbb{R}^n$ is non-parabolic if and only if $n > 2$.

Moreover, notice that on Ψ_r

$$\frac{\partial G_\Omega}{\partial \nu} = N \cdot \nabla G_\Omega = -|\nabla G_\Omega| \ , \tag{3.2.4}$$

is valid, where N is the normal vector. Now, by Stokes' theorem

$$\int\limits_{\Psi_r} \frac{\partial v}{\partial \nu} G_\Omega dA = \frac{1}{r^a} \int\limits_{\Psi_r} \frac{\partial v}{\partial \nu} dA = \frac{1}{r^a} \int\limits_{\Omega_r} \Delta v d\mu \ . \tag{3.2.5}$$

By combining (3.2.3), (3.2.4), (3.2.5) we have

$$\begin{aligned} v(x) &= -\int\limits_{\Psi_r} (\frac{\partial G_\Omega}{\partial \nu} v - \frac{\partial v}{\partial \nu} G_\Omega) dA - \int\limits_{\Omega_r} G_\Omega \Delta v d\mu \\ &= -\int\limits_{\Psi_r} (\frac{\partial G_\Omega}{\partial \nu} v) dA + \int\limits_{\Psi_r} \frac{\partial v}{\partial \nu} G_\Omega dA - \int\limits_{\Omega_r} G_\Omega \Delta v d\mu \\ &= \int\limits_{\Psi_r} |\nabla G_\Omega| v dA + \frac{1}{r^a} \int\limits_{\Omega_r} \Delta v d\mu - \int\limits_{\Omega_r} G_\Omega \Delta v d\mu \\ &= \int\limits_{\Psi_r} |\nabla G_\Omega| v dA - \int\limits_{\Omega_r} (G_\Omega - r^{-a}) \Delta v d\mu \\ &= \int\limits_{\Psi_r} |\nabla G_\Omega| v dA - \int\limits_{\Omega_r} \phi_r \Delta v d\mu \end{aligned}$$

which is (3.2.2).

Equality (3.2.1) follows now from (3.2.2) by the co-area[4] formula. Indeed, multiplying with η^{a-1} on the both sides of (3.2.2) and integrate on $[0, r]$, we have that

$$\int\limits_0^r \eta^{a-1} v(x) d\mu = \int\limits_0^r \eta^{a-1} \{ \int\limits_{\Psi_r} |\nabla G_\Omega| v dA - \int\limits_{\Omega_r} \phi_r \Delta v d\mu \} \ .$$

Hence, by straightforward computations we get

$$\begin{aligned} \frac{1}{a} r^a v(x) &= \int\limits_0^r \eta^{a-1} \int\limits_{\Psi_r} |\nabla G_\Omega| v dA d\eta - \int\limits_0^r \eta^{a-1} \int\limits_{\Omega_r} \phi_r \Delta v d\mu d\eta \\ &= \int\limits_0^r \eta^{a-1} \int\limits_{\Psi_\eta} |\nabla G_\Omega| v dA d\eta - \int\limits_0^r \eta^a \int\limits_{\Omega_\eta} \phi_\eta \Delta v d\mu \frac{d\eta}{\eta} \end{aligned}$$

[4]For further details see Appendix 6.2: The Co-area formula on Riemannian manifolds

$$= \frac{1}{a}\int\limits_{\infty}^{r^{-a}} \frac{1}{\beta^2} \int\limits_{G_\Omega=\beta} |\nabla G_\Omega| v dA da - \int\limits_0^r \eta^a \int\limits_{\Omega_\eta} \phi_\eta \Delta v d\mu \frac{d\eta}{\eta}$$

$$= \frac{1}{a}\int\limits_{r^{-a}}^{\infty} \frac{1}{\beta^2} \int\limits_{G_\Omega=\beta} |\nabla G_\Omega| v dA da - \int\limits_0^r \eta^a \int\limits_{\Omega_\eta} \phi_\eta \Delta v d\mu \frac{d\eta}{\eta}$$

$$= \frac{1}{a}\int\limits_{r^{-a}}^{\infty} \int\limits_{G_\Omega=\beta} \frac{|\nabla G_\Omega|}{|G_\Omega^2|} v dA da - \int\limits_0^r \eta^a \int\limits_{\Omega_\eta} \phi_\eta \Delta v d\mu \frac{d\eta}{\eta}$$

$$= \frac{1}{a}\int\limits_{r^{-a}}^{\infty} \int\limits_{G_\Omega=\beta} \frac{|\nabla \log G_\Omega|^2}{|\nabla G_\Omega|} v dA da - \int\limits_0^r \eta^a \int\limits_{\Omega_\eta} \phi_\eta \Delta v d\mu \frac{d\eta}{\eta}\ .$$

Now, we apply the co-area formula to obtain

$$\frac{1}{a} r^a v(x) = \frac{1}{a} \int\limits_{G_\Omega \geq \beta} |\nabla \log G_\Omega|^2 v d\mu - \int\limits_0^r \eta^a \int\limits_{\Omega_\eta} \phi_\eta \Delta v d\mu \frac{d\eta}{\eta}$$

$$= \frac{1}{a} \int\limits_{\Omega_r} |\nabla \log G_\Omega|^2 v d\mu - \int\limits_0^r \eta^a \int\limits_{\Omega_\eta} \phi_\eta \Delta v d\mu \frac{d\eta}{\eta}\ .$$

Hence we have,

$$v(x) = \frac{a}{r^a}\frac{1}{a} \int\limits_{\Omega_r} |\nabla \log G_\Omega|^2 v d\mu - \frac{a}{r^a} \int\limits_0^r \eta^a \int\limits_{\Omega_\eta} \phi_\eta \Delta v d\mu \frac{d\eta}{\eta}\ ,$$

which implies

$$v(x) = \frac{1}{r^a} \int\limits_{\Omega_r} |\nabla \log G_\Omega|^2 v d\mu - \frac{a}{r^a} \int\limits_0^r \eta^a \int\limits_{\Omega_\eta} \phi_\eta \Delta v d\mu \frac{d\eta}{\eta}\ .$$

□

Since we have applied the co-area formula, we have to verify that $|\nabla \log G_\Omega|^2$ is integrable. This follows from the asymptotic behavior of $G_\Omega(x,y)$ near x. Also, a gradient estimate by Cheng-Yau (see [8], Theorem 6.1) asserts that near x,

$$|\nabla \log G_\Omega|^2(y) \leq k\left(1 + \frac{1}{d^2(x,y)}\right)\ ,$$

for some constant $k > 0$. We have the following global result.

Definition 3.2.2 *The Riemannian manifold (M,g) is called strongly non-parabolic if*

$$\lim_{y\to\infty} G(x,y) = 0 \ ,$$

where G is the minimal positive Green's fuction on M.

We now define, similarly to the bounded domain Ω before, the useful quantities: G-sphere, G-ball, ϕ_r and the function $\psi_r = \log(Gr^n)$, global on M.

Theorem 3.2.3 *Assume that (M,g) is a strongly non-parabolic Riemannian manifold. Let v be a smooth function (as in Proposition 3.2.1). Then for every $r > 0$*

$$v(x) = \frac{1}{r^a}\int\limits_{\Omega_r} |\nabla \log G_\Omega|^2 v d\mu - \int\limits_0^r \frac{a}{\eta^{a+1}} \int\limits_{\Omega_\eta} \psi_\eta \Delta v d\mu d\eta. \tag{3.2.6}$$

As before, for almost every $r > 0$ we have the same expression for $v(x)$, i.e.

$$v(x) = \int\limits_{\Psi_r} |\nabla G| v dA_y - \int\limits_{\Omega_r} \phi_r \Delta v d\mu_y. \tag{3.2.7}$$

Proof. The only thing we need to verify is that

$$\int\limits_0^r \frac{a}{\eta^{a+1}} \int\limits_{\Omega_\eta} \psi_\eta \Delta v d\mu d\eta = \frac{a}{r^a}\int\limits_0^r \eta^a \int\limits_{\Omega_\eta} \phi_\eta \Delta v d\mu \frac{d\eta}{\eta}. \tag{3.2.8}$$

To prove this, we will use Tonelli's theorem together with some straightforward computations. Thus we have, on the right-hand side of (3.2.8), that

$$\begin{aligned}
\int\limits_0^r \eta^a \int\limits_{\Omega_\eta} \phi_\eta \Delta v d\mu \frac{d\eta}{\eta} &= \int\limits_0^r \eta^{a-1} \int\limits_{\Omega_\eta} (G - \eta^{-a}) \Delta v d\mu d\eta = \int\limits_0^r \int\limits_{\Omega_\eta} (G\eta^{a-1} - \eta^{-1}) \Delta v d\mu d\eta \\
&= \int\limits_0^r \int\limits_{G \geq \eta^{-a}} (G\eta^{a-1} - \eta^{-1}) \Delta v d\mu d\eta \\
&= \int\limits_{G \geq r^{-a}} \int\limits_{G^{-\frac{1}{a}}}^r (G\eta^{a-1} - \eta^{-1}) \Delta v d\mu d\eta \\
&= \int\limits_{\Omega_r} [G\frac{\eta^a}{a} - \log\eta]_{G^{-\frac{1}{a}}}^r \Delta v d\mu \\
&= \int\limits_{\Omega_r} [G\frac{r^a}{a} - \log r - \frac{1}{a} + \log G^{\frac{-1}{a}}] \Delta v d\mu
\end{aligned}$$

$$= \frac{r^a}{a} \int_{\Omega_r} (G - \frac{1}{r^a}) \Delta v d\mu - \int_{\Omega_r} \frac{1}{a}(a \log r + \log G) \Delta v d\mu$$

$$= \frac{r^a}{a} \int_{\Omega_r} \phi_r \Delta v d\mu - \frac{1}{a} \int_{\Omega_r} \log(G r^a) \Delta v d\mu \ . \tag{3.2.9}$$

We now apply Tonelli's theorem on the left-hand side of (3.2.8) :

$$\int_0^r \frac{a}{\eta^{a+1}} \int_{\Omega_\eta} \psi_\eta \Delta v d\mu d\eta = a \int_0^r \int_{G \geq \eta^{-a}} (\eta^{-a-1} \psi_\eta) \Delta v d\mu d\eta$$

$$= a \int_{G \geq r^{-a}} \int_{G^{-\frac{1}{a}}}^r (\eta^{-a-1} \psi_\eta) \Delta v d\eta d\mu$$

$$= a \int_{G \geq r^{-a}} [\int_{G^{-\frac{1}{a}}}^r \eta^{-a-1} \log G d\eta + \int_{G^{-\frac{1}{a}}}^r \eta^{-a-1} \log \eta^a d\eta] \Delta v d\mu$$

$$= a \int_{G \geq r^{-a}} ([\log G \frac{\eta^{-a}}{-a}]^r_{G^{-\frac{1}{a}}} + (-\eta^{-a} \log \eta - \frac{\eta^{-a}}{a}) \Delta v d\mu$$

$$= a \int_{\Omega_r} \frac{1}{a}(-r^{-a} \log G - a r^{-a} \log r - r^{-a} + G) \Delta v d\mu$$

$$= \int_{\Omega_r} [(G - r^{-a}) - (r^{-a} \log G + a r^{-a} \log r)] \Delta v d\mu$$

$$= \int_{\Omega_r} (\Delta_v) \phi_r d\mu - \int_{\Omega_r} r^{-a} (\log G + \log r^a) \Delta v d\mu$$

$$= \int_{\Omega_r} (\Delta_v) \phi_r d\mu - r^{-a} \int_{\Omega_r} (\Delta_v) \psi_r d\mu \ . \tag{3.2.10}$$

From (3.2.8), (3.2.9) and (3.2.10) we have,

$$\frac{r^a}{a} \int_{\Omega_r} (\Delta_v) \phi_r d\mu - \frac{1}{a} \int_{\Omega_r} (\Delta_v) \psi_r d\mu = \frac{r^a}{a} [\int_{\Omega_r} (\Delta_v) \phi_r d\mu - \int_{\Omega_r} r^{-a} (\log G + \log r^a) \Delta v d\mu] \tag{3.2.11}$$

$$= \frac{r^a}{a} \int_{\Omega_r} (\Delta_v) \phi_r d\mu - \frac{1}{a} \int_{\Omega_r} (\Delta_v) \psi_r d\mu \ .$$

□

Corollary 3.2.4 *For any smooth (or Lipschitz) function v, we define*

$$J_v(r) := \int_{\Psi_r} |\nabla G| v dA \ ,$$

with Ψ_r and G as before. Then for almost every $r > 0$ and positive a we have

$$\frac{d}{dr} J_v(r) = \frac{a}{r^{a+1}} \int_{\Omega_r} \Delta v d\mu \ .$$

Proof. We have from the Proposition 3.2.1 that (3.2.7) holds globally on M. We differentiate (3.2.7) with respect to r :

$$\begin{aligned} v(x) =& \int_{\Psi_r} |\nabla G| v dA_y - \int_{\Omega_r} \phi_r \Delta v d\mu_y \\ \Rightarrow& v(x) = J_v(r) - \int_{\Omega_r} \phi_r \Delta v d\mu_y \\ \Rightarrow& 0 = \frac{d}{dr}[J_v(r)] - \frac{d}{dr}[\int_{\Omega_r} \phi_r \Delta v d\mu_y] \ , \end{aligned}$$

and we obtain

$$\begin{aligned} J_v'(r) =& \frac{d}{dr}[\int_{\Omega_r} (G - r^{-a}) \Delta v d\mu] = \frac{d}{dr}[\int_{\Omega_r} G \Delta v d\mu - \frac{1}{r^a} \int_{\Omega_r} \Delta v d\mu] = \\ =& \frac{d}{dr}[\int_0^r d\eta \int_{G^{-\frac{1}{a}}=\eta} \frac{G\Delta v}{|\nabla G^{-\frac{1}{a}}|} dA - \frac{1}{r^a} \int_0^r d\eta \int_{G^{-\frac{1}{a}}=\eta} \frac{\Delta v}{|\nabla G^{-\frac{1}{a}}|} dA] \\ =& \int_{G^{-\frac{1}{a}}=r} \frac{G\Delta v}{|\nabla G^{-\frac{1}{a}}|} dA - \frac{1}{r^a} \int_{G^{-\frac{1}{a}}=r} \frac{\Delta v}{|\nabla G^{-\frac{1}{a}}|} dA + \frac{ar^{a-1}}{r^2 a} \int_0^r d\eta \int_{G^{-\frac{1}{a}}=\eta} \frac{G\Delta v}{|\nabla G^{-\frac{1}{a}}|} dA \ . \end{aligned}$$

The first two orders in the last equality are equal since $G = r^{-a}$ and by the co-area formula again, we have the desired result.

□

Chapter 4

New Monotonicity Formulas for Ricci Curvature and Applications

4.1 The first monotonicity formula

Let (M^n, g) be a complete Riemannian manifold with $\dim(M) = n \geq 3$. For now, we will not require any conditions about the Ricci curvature of M.

We now define the function b, via Green's function, and this function which will be used as a generalized distance function. Thus, for the function $b = G^{\frac{1}{2-n}}$, we have that

$$\nabla b^2 = \left(\frac{2}{2-n}\right) G^{\frac{2}{2-n}} \nabla G \ .$$

Since for any smooth function f and for every $n \in \mathbb{Z}$, we have that

$$\Delta f^n = n(n-1)^{n-2}|\nabla f|^2 + nf^{n-1}\Delta f,$$

then,

$$\Delta b^2 = \frac{2n}{(2-n)^2} G^{\frac{2n-2}{2-n}} |\nabla G|^2 + \left(\frac{2}{2-n}\right) G^{\frac{n}{2-n}} \Delta G \ . \tag{4.1.1}$$

Let $\phi \in C^\infty(M)$ be a function with compact support. Then we have that

$$\int_M \Delta G_x(y)\ G^{\frac{2}{2-n}}(y)\ \phi(y)\ d\mu_y = 0\ ,$$

since $\frac{2}{2-n} < 0$ and $G^{\frac{2}{2-n}}$ is a continous function on M with

$$G^{\frac{2}{2-n}}(x) = 0\ .$$

The last in turn implies that

$$G^{\frac{n}{2-n}} \Delta G = 0\ ,$$

which together with (4.1.1) give us the following useful relation

$$\Delta b^2 = 2n|\nabla b|^2 \ . \tag{4.1.2}$$

Next we consider a quantity which we will see several times in what follows. That is an integral over the level sets of the function b, defined as

$$I_v(r) = r^{1-n} \int_{b=r} v|\nabla b| dArea = \frac{1}{n-2} \int_{b=r} v|\nabla G| dArea \ ,$$

where v is a smooth function on M. Then we have from Corollary 3.2.4 of Chapter 3, that for $a = n-2 > 0$,

$$\frac{d}{dr}(I_v(r)) = I_v'(r) = r^{1-n} \int_{b=r} v_n dArea \ , \tag{4.1.3}$$

where v_n: is the outward normal derivative of the function v, normal to the boundary of $\{x : b(x) \leq r\}$. From Stokes' theorem, the right hand side of (4.1.3) is equal to

$$r^{1-n} \int_{b \leq r} \Delta v d\text{Vol} \ .$$

4.1.1 Normalized Generalized Area and Volume

In this section we introduce two new quantities on M, the normalized generalized area $A(r)$ and volume $V(r)$ of balls with radius r. We define the non-negative functions

$$A(r) = r^{1-n} \int_{b=r} |\nabla b|^3 dArea$$

and

$$V(r) = r^{-n} \int_{b \leq r} |\nabla b|^4 d\text{Vol} \ .$$

We will use the results of the next Lemma to understand the behavior of the normalized generalized area and volume and the function b, as r tends to 0.

Lemma 4.1.1 *Let M^n be a smooth manifold with $n \geq 3$. Then for $r \to 0$ the function $b(x)$ behaves like the radius of balls (or Euclidean distance). In other words, in small balls, the following properties hold:*

$$\text{i)} \quad \lim_{r\to 0} \sup_{\partial B_r(x)} \left|\frac{b}{r} - 1\right| = 0$$

$$\text{ii)} \quad \lim_{r\to 0} \sup_{\partial B_r(x)} \left||\nabla b|^2 - 1\right| = 0$$

$$\text{iii)} \quad \lim_{r\to 0} A(r) = \lim_{r\to 0} I_1(r) = \lim_{r\to 0} \mathrm{Vol}(\partial B_1(0))$$
$$\text{iv)} \quad \lim_{r\to 0} V(r) = \mathrm{Vol}(B_1(0)) \ ,$$

where $I_1(r)$ in (iii), *is the function*

$$I_1(r) = r^{1-n} \int\limits_{b=r} |\nabla b| dArea$$

which is constant in the level set $b = r$, as function with respect to r.

Proof. For the first two claims $(i), (ii)$ we will use a result of a paper of Gilbarg-Serrin ([24]), in which some estimates for the Green function were proven. Hence we have that for a Green function $G(x, y)$ with pole at x, holds

$$G_x(y) = d^{2-n}(x, y)(1 + o(1))$$

and for its gradient

$$|\nabla G_x(y)| = (n-2)d^{1-n}(x, y)(1 + o(1)) \ ,$$

where $o(1)$ is a function with $o(y) \to 0$ as $y \to x$. Now it is easy to see that, for $b = G^{\frac{1}{2-n}}$ it follows

$$\begin{aligned} b &= d(x, y)(1 + o(1))^{\frac{1}{2-n}} \\ &= d(x, y)(1 + o(1)), \end{aligned}$$

Therefore

$$\lim_{r\to 0} \sup_{\partial B_r(x)} \Big|\frac{d(1 + o(1))}{d} - 1\Big| = 0 \ ,$$

which proves (i).

We accordingly have for the second claim,

$$|\nabla b|^2 = (1 + o(1)) \ ,$$

since

$$|\nabla G| = (n-2)b^{1-n}|\nabla b|$$

and hence

$$\lim_{r\to 0} \sup_{\partial B_r(x)} \big||\nabla b|^2 - 1\big| = 0 \ .$$

For the two other claims, we first have to observe that

$$\lim_{r\to 0} I_1(r) = \lim_{r\to 0} \left[r^{1-n} \int\limits_{b=r} |\nabla b| dArea \right]$$

$$= \lim_{r\to 0}\left[r^{1-n}\int\limits_{b=r}|\nabla b|^3 dArea\right]$$
$$= \lim_{r\to 0} A(r) \ ,$$

from (ii). Moreover by the coarea formula we have that

$$\int\limits_{b\le r}|\nabla b|^2 d\mathrm{Vol} = \int\limits_{-\infty}^{+\infty}\int\limits_{b=s}|\nabla b| \ dArea \ ds$$
$$= \int\limits_0^r s^{n-1} I_1(s) \ ds$$
$$= \frac{r^n}{n} I_1(r)$$

and thus $I_1(r)$ can be written as

$$I_1(r) = \frac{n}{r^n}\int\limits_{b\le r}|\nabla b|^2 d\mathrm{Vol} \ .$$

Finally, for the limit of $I_1(r)$ we have

$$\lim_{r\to 0} I_1(r) = I_1(1) = \lim_{r\to 0}\frac{n}{r^n}\int\limits_{b\le r}|\nabla b|^2 d\mathrm{Vol} \ . \tag{4.1.4}$$

Now, we observe that for $r \to 0$, we have

$$\frac{\int\limits_{b\le r} d\mathrm{Vol}}{\mathrm{Vol}(B_0(r))} \to 1 \ ,$$

where $\mathrm{Vol}(B_0(r)) = \mathrm{Vol}(B_0(1))r^n$ is the Euclidean volume for the ball $B_x(r)$, centered at some x.

To see this, first note that (i) means that for $\delta > 0$ and for r sufficiently small, we have

$$\frac{r}{1+\delta} \le b \le \frac{r}{1-\delta} \ .$$

Therefore,

$$B_x(t(1-\delta)) \subseteq \{b \le t\} \subseteq B_x(t(1+\delta)) \ ,$$

and then

$$\frac{\left|\mathrm{Vol}(\{b\le t\}) - \mathrm{Vol}(B_x(t))\right|}{\left|\mathrm{Vol}(B_x(t))\right|} \le \frac{\left|\mathrm{Vol}[B_x(t(1+\delta))] - \mathrm{Vol}[B_x(t(1-\delta))]\right|}{\left|\mathrm{Vol}(B_x(t))\right|} \ ,$$

where the left-hand side of the above inequality tends to $(1+\delta)^n - (1-\delta)^n = \delta(2n + o(1))$. Finally,

$$\mathrm{Vol}(\{b \le t\}) = \mathrm{Vol}(B_x(t))(1 + o(1)) ,$$

as $t \to 0$ and since

$$\mathrm{Vol}(B_x(t)) = \mathrm{Vol}(B_0(1))(1 + o(1)) ,$$

for $t \to 0$, we conclude our claim.

Hence, the left hand side of (4.1.4) is equal to $\mathrm{Vol}(\partial B_1(0))$ and as $|\nabla b|^4$ tends to 1 as well as $r \to 0$,

$$\lim_{r\to 0} V(r) = \mathrm{Vol}(B_1(0)) ,$$

which is (iv).

□

Before studying the first monotonicity formula, we need two more Lemmas:

Lemma 4.1.2 *For the functions $A(r)$ and $V(r)$, as described before, we have*

$$V'(r) = \frac{1}{r}[A(r) - nV(r)] .$$

Proof. By the coarea formula again, we can rewrite $V(r)$ as

$$\begin{aligned} V(r) &= r^{-n} \int\limits_{b\le r} |\nabla b|^4 dVol = r^{-n} \int\limits_0^r \int\limits_{b=t} |\nabla b|^3 dArea\ dt \\ &= r^{-n} \int\limits_0^r t^{n-1} A(t)\ dt . \end{aligned}$$

Then for the derivative of $V(r)$ with respect to r we have

$$\begin{aligned} V'(r) &= -nr^{-n-1} \int\limits_0^r t^{n-1} A(t)\ dt + r^{-n} r^{n-1} A(r) \\ &= -nr^{-1} V(r) + r^{-n} r^{n-1} A(r) \\ &= \frac{1}{r}[A(r) - nV(r)] . \end{aligned}$$

□

Lemma 4.1.3 *If the gradient of b is bounded from above by a constant C depending only on the dimension of M^n, i.e. $|\nabla b| \le C(n) = C$, then the generalized area $A(r)$ and volume $V(r)$ are bounded by their Euclidean values (their values as $r \to 0$) multiplied by a constant.*

Proof. We will show that

$$\begin{aligned} &i) \;\; A \leq C^2 \mathrm{Vol}(\partial B_1(0)) \;\; \textit{and} \\ &ii) \;\; V \leq C^2 \mathrm{Vol}(B_1(0)) \; . \end{aligned}$$

The first claim follows from (*iii*) of Lemma 4.1.1, together with the fact that $I_1(r)$ is constant as a function of r and the bound for $|\nabla b|$.

For the second claim we have from the alternative expression of $V(r)$:

$$V(r) = r^{-n} \int_0^r t^{n-1} A(t) \; dt \; ,$$

and from (*i*) that

$$V \leq C^2 \frac{\mathrm{Vol}(\partial B_1(0))}{n} = C^2 \mathrm{Vol}(B_1(0)) \; .$$

□

Now we are ready to state our first monotonicity formula, in the following theorem. This monotonicity formula, together with two other that follow, are the intermediate results of the study of the monotonicity of $A(r)$ and $V(r)$, which we are of our interested.

Before this, we first cite an auxiliary result from Geometric Analysis, the Bochner-Weitzenbock formula[1], which relates harmonic functions on a Riemannian manifold (M^n, g) to the Ricci curvature. From this very important technique, which is used here for the function b^2, we get the following identity

$$\frac{1}{2}\Delta |\nabla b^2|^2 = |\mathrm{Hess}_{b^2}|^2 + \langle \nabla \Delta b^2, \nabla b^2 \rangle + \mathrm{Ric}(\nabla b^2, \nabla b^2) \; .$$

Theorem 4.1.4 *On a Riemannian manifold (M^n, g), $n \geq 3$, with A and V to be the 'area' and 'volume' of balls respectively, one has*

$$[A(r) - 2(n-1)V(r)]'(r) = \frac{r^{-1-n}}{2} \int_{b \leq r} \left(\left|\mathrm{Hess}_{b^2} - \frac{\Delta b^2}{n} g\right|^2 + \mathrm{Ric}(\nabla b^2, \nabla b^2) \right) d\mathrm{Vol} \; . \tag{4.1.5}$$

In particular, when the Ricci curvature is non-negative, the function

$$F(r) = A(r) - 2(n-1)V(r)$$

is non-decreasing in r.

[1] For more details about the theorem and its proof, see the Appendix 6.1: Bochner-Weitzenbock formula

Proof. At first, we note that generally the right side of the equation (4.1.5) has no sign, and this is obtained when the Ricci curvature is non-negative.

We now observe that we can rewrite $A(r)$ as

$$A(r) = r^{1-n} \int\limits_{b=r} |\nabla b|^2 |\nabla b| dArea = r^{1-n} \int\limits_{b=r} \frac{|\nabla b^2|^2}{4b^2} |\nabla b| dArea$$
$$= \frac{r^{-1-n}}{4} \int\limits_{b=r} |\nabla b^2|^2 |\nabla b| dArea \ ,$$

since $|\nabla b^2|^2 = 4b^2|\nabla b|^2$ and $b \neq 0$, for $r > 0$.

We will now 'built' the expression we need. We have that

$$r^2 A(r) = \frac{r^{1-n}}{4} \int\limits_{b=r} |\nabla b^2|^2 |\nabla b| dArea$$

and the derivation of this equality gives

$$\begin{aligned} r^{-2}[r^2 A(r)]'(r) &= r^{-2} \frac{d}{dr} \left(\frac{r^{1-n}}{4} \int\limits_{b=r} |\nabla b^2|^2 |\nabla b| dArea \right) \\ &= r^{-2} \left[\frac{r^{1-n}}{4} \int\limits_{b=r} \frac{\partial}{\partial n} (|\nabla b^2|^2) dArea \right] \\ &= \frac{r^{-1-n}}{4} \int\limits_{b=r} \frac{\partial}{\partial n} (|\nabla b^2|^2) dArea \\ &= \frac{r^{-1-n}}{4} \int\limits_{b \leq r} \Delta(|\nabla b^2|^2) d\text{Vol} \ , \end{aligned} \tag{4.1.6}$$

where the expression into the brackets in (4.1.6) came just as the integral in (4.1.3) for $v = (|\nabla b^2|^2)$. Now continuing, from the Bochner-Weitzenbock formula it follows

$$\frac{r^{-1-n}}{2} \int\limits_{b \leq r} \left[|\text{Hess}_{b^2}|^2 + \langle \nabla \Delta b^2, \nabla b^2 \rangle + \text{Ric}(\nabla b^2, \nabla b^2) \right] d\text{Vol}$$
$$= \frac{r^{-1-n}}{2} \int\limits_{b \leq r} \left[|\text{Hess}_{b^2}|^2 - |\Delta b^2|^2 + \text{Ric}(\nabla b^2, \nabla b^2) \right] d\text{Vol} \ +$$
$$+ \ \frac{r^{-1-n}}{2} \int\limits_{b=r} \Delta b^2 \langle \nabla b^2, N \rangle dArea$$

$$=\frac{r^{-1-n}}{2}\int\limits_{b\le r}\left[|\mathrm{Hess}_{b^2}|^2-|\Delta b^2|^2+\mathrm{Ric}(\nabla b^2,\nabla b^2)\right]d\mathrm{Vol}\ +\ \frac{r^{-1-n}}{2}\int\limits_{b=r}4nb|\nabla b|^3 dArea$$
$$=\frac{r^{-1-n}}{2}\int\limits_{b\le r}\left[|\mathrm{Hess}_{b^2}|^2-|\Delta b^2|^2+\mathrm{Ric}(\nabla b^2,\nabla b^2)\right]d\mathrm{Vol}\ +\ \frac{2nr^{1-n}}{r}\int\limits_{b=r}|\nabla b|^3 dArea$$
$$=\frac{r^{-1-n}}{2}\int\limits_{b\le r}\left[|\mathrm{Hess}_{b^2}|^2-|\Delta b^2|^2+\mathrm{Ric}(\nabla b^2,\nabla b^2)\right]d\mathrm{Vol}\ +\ \frac{2n}{r}A(r)\ , \tag{4.1.7}$$

from the Divergence Theorem, since the unit normal to the hypersurface $\{b=r\}$ is $N=\frac{\nabla b}{|\nabla b|}$.

Note that, if X is a compactly supported vector field on $U\subseteq M$ with boundary ∂U, then the Divergence Theorem[2] states that

$$\int\limits_U \mathrm{div}X\,d\mathrm{Vol}=\int\limits_{\partial U}\langle X,n\rangle dArea\ ,$$

where n is the outward-pointing normal on ∂U.

Moreover, since

$$\begin{aligned}\left|\mathrm{Hess}_{b^2}-\frac{\Delta b^2}{n}g\right|^2&=\left|\mathrm{Hess}_{b^2}\right|^2-2\frac{\Delta b^2}{n}\langle g,\mathrm{Hess}_{b^2}\rangle+\frac{\left|\Delta b^2\right|^2}{n^2}\langle g,g\rangle\\&=\left|\mathrm{Hess}_{b^2}\right|^2-\frac{\left|\Delta b^2\right|^2}{n}\ ,\end{aligned}\tag{4.1.8}$$

we have that

$$\begin{aligned}\left|\mathrm{Hess}_{b^2}\right|^2-\left|\Delta b^2\right|^2&=\left|\mathrm{Hess}_{b^2}-\frac{\Delta b^2}{n}g\right|^2-\left(1-\frac{1}{n}\right)\left|\Delta b^2\right|^2\\&=\left|\mathrm{Hess}_{b^2}-\frac{\Delta b^2}{n}g\right|^2-4n^2\left(1-\frac{1}{n}\right)\left|\nabla b\right|^4\ .\end{aligned}$$

After these computations, (4.1.7) becomes

$$\begin{aligned}&\frac{r^{-1-n}}{2}\int\limits_{b\le r}\left(\left|\mathrm{Hess}_{b^2}-\frac{\Delta b^2}{n}g\right|^2+\mathrm{Ric}(\nabla b^2,\nabla b^2)\right)d\mathrm{Vol}-\\&\qquad-2\left(1-\frac{1}{n}\right)n^2r^{-1-n}\int\limits_{b\le r}|\nabla b|^4 d\mathrm{Vol}+\frac{2n}{r}A(r)\\&=\ \frac{r^{-1-n}}{2}\int\limits_{b\le r}\left(\left|\mathrm{Hess}_{b^2}-\frac{\Delta b^2}{n}g\right|^2+\mathrm{Ric}(\nabla b^2,\nabla b^2)\right)d\mathrm{Vol}-\frac{2n^2}{r}\left(1-\frac{1}{n}\right)V(r)\end{aligned}$$

[2] See ([15], 1.2).

$$+\frac{2n}{r}A(r) \ .$$

Now setting I for the first term and using Lemma 4.1.2, the last is equal to

$$\begin{aligned} &I-\frac{2n^2}{r}V(r)+\frac{2n}{r}V(r)+\frac{2n}{r}A(r)\\ =&\frac{2n}{r}(A(r)-nV(r))+\frac{2n}{r}V(r) \ . \end{aligned}$$

Finally, since $A'(r)=r^{-2}(r^2A)'(r)-\frac{2}{r}A(r)$ we have that

$$\begin{aligned} A'(r)&=I+\frac{2n}{r}(A(r)-nV(r))+\frac{2n}{r}V(r)-\frac{2}{r}A(r)\\ &=\frac{2(n-1)}{r}\big[A(r)-nV(r)\big] \end{aligned}$$

and then

$$\begin{aligned} \big[A(r)-nV(r)\big]'(r)&=A'(r)-2(n-1)V'(r)\\ &=I+\frac{2(n-1)}{r}\big[A(r)-nV(r)\big]-2(n-1)V'(r)\\ &=I \ , \end{aligned}$$

or equivalently,

$$\big[A(r)-nV(r)\big]'(r)=\frac{r^{-1-n}}{2}\int\limits_{b\le r}\left(\left|\mathrm{Hess}_{b^2}-\frac{\Delta b^2}{n}g\right|^2+\mathrm{Ric}(\nabla b^2,\nabla b^2)\right)d\mathrm{Vol} \ .$$

□

Note that we have not need any assumptions for the Ricci curvature in our theorem. But now we have the following result on a manifold with non-negative Ricci curvature.

Corollary 4.1.5 *If M is an n-dimensional manifold with non-negative Ricci curvature, then, for all $r>0$, we have*

$$A(r)-\mathrm{Vol}(\partial B_1(0))\ge 2(n-1)\big[V(r)-\mathrm{Vol}(B_1(0))\big] \ .$$

Moreover, if for some $r>0$ we have equality, then the set $\{x : b(x)\le r\}$ is isometric to a ball of radius r in $\mathbb{R}^n$.

Proof. Since F is non-decreasing with respect to r, we have for $s<r$ that

$$F(s)=A(s)-2(n-1)V(s)\le A(r)-2(n-1)V(r)=F(r) \ .$$

Taking the limit on the left side of inequality, with $s\to 0$, we have from Lemma 4.1.1, that

$$\mathrm{Vol}(\partial B_1(0))-2(n-1)\mathrm{Vol}(B_1(0))\le A(r)-2(n-1)V(r) \ .$$

But this gives us the expected, i.e.

$$A(r)-\mathrm{Vol}(\partial B_1(0))\ge 2(n-1)\big[V(r)-\mathrm{Vol}(B_1(0))\big] \ .$$

□

4.2 The second Monotonicity Formula

The next Lemma is useful in the proof of the second monotonicity formula, which lies in the next theorem. Note that this Lemma holds for any positive harmonic function G, where, as before, b is given by $b^{2-n} = G$.

Lemma 4.2.1 *The following identities hold:*

$$b^2\Delta|\nabla b|^2 + (2-n)\langle\nabla b^2, \nabla|\nabla b|^2\rangle = \frac{1}{2}\left(\left|\mathrm{Hess}_{b^2} - \frac{\Delta b^2}{n}g\right|^2 + \mathrm{Ric}(\nabla b^2, \nabla b^2)\right) ,$$

$$\Delta(|\nabla b|^2 G) = \frac{1}{2}\left(\left|\mathrm{Hess}_{b^2} - \frac{\Delta b^2}{n}g\right|^2 + \mathrm{Ric}(\nabla b^2, \nabla b^2)\right) b^{-n} .$$

Proof. By the Bochner-Weitzenbock formula again and (4.1.8), we have

$$\begin{aligned}\frac{1}{2}\Delta|\nabla b^2|^2 &= |\mathrm{Hess}_{b^2}|^2 + \langle\nabla\Delta b^2, \nabla b^2\rangle + \mathrm{Ric}(\nabla b^2, \nabla b^2)\\ &= \left|\mathrm{Hess}_{b^2} - \frac{\Delta b^2}{n}g\right|^2 + \frac{|\Delta b^2|^2}{n} + \langle\nabla\Delta b^2, \nabla b^2\rangle + \mathrm{Ric}(\nabla b^2, \nabla b^2)\\ &= \left|\mathrm{Hess}_{b^2} - \frac{\Delta b^2}{n}g\right|^2 + 4n|\nabla b|^4 + 2n\langle\nabla|\nabla b|^2, \nabla b^2\rangle + \mathrm{Ric}(\nabla b^2, \nabla b^2) ,\end{aligned} \tag{4.2.1}$$

since of course $|\Delta b^2|^2 = 4n^2|\nabla b|^4$. Moreover, from the fact that

$$|\nabla b^2|^2 = 4b^2|\nabla b|^2 ,$$

it follows

$$\begin{aligned}\Delta|\nabla b^2|^2 &= 4b^2\Delta|\nabla b|^2 + 4(\Delta b^2)|\nabla b|^2 + 8\langle\nabla b^2, \nabla|\nabla b|^2\rangle\\ &= 4b^2\Delta|\nabla b|^2 + 8n|\nabla b|^4 + 8\langle\nabla b^2, \nabla|\nabla b|^2\rangle .\end{aligned} \tag{4.2.2}$$

For the first claim we have from (4.2.1) and (4.2.2) that

$$\begin{aligned}\left|\mathrm{Hess}_{b^2} - \frac{\Delta b^2}{n}g\right|^2 + \mathrm{Ric}(\nabla b^2, \nabla b^2) &= \frac{1}{2}\Delta|\nabla b^2|^2 - 4n|\nabla b|^4 - 2n\langle\nabla b^2, \nabla|\nabla b|^2\rangle\\ &= \frac{1}{2}(4b^2\Delta|\nabla b|^2 + 8n|\nabla b|^4 + 8\langle\nabla b^2, \nabla|\nabla b|^2\rangle)\\ &\qquad - 4n|\nabla b|^4 - 2n\langle\nabla b^2, \nabla|\nabla b|^2\rangle\\ &= 2b^2\Delta|\nabla b|^2 + 2(2-n)\langle\nabla b^2, \nabla|\nabla b|^2\rangle .\end{aligned}$$

The second claim follows easily from the fact that for any two smooth functions f, g we have $\Delta(fg) = \Delta f\ g + 2\langle\nabla f, \nabla g\rangle + \Delta g\ f$. This together with the first claim, gives

$$2\Delta(|\nabla b|^2 G) = 2\ \Delta|\nabla b|^2 G + 4\langle\nabla G, \nabla|\nabla b|^2\rangle + 2|\nabla b|^2\Delta G$$

$$
\begin{aligned}
&= 2\ \Delta|\nabla b|^2 G + 4\langle \nabla G, \nabla|\nabla b|^2\rangle \\
&= 2\ \Delta|\nabla b|^2 b^{2-n} + b^{-n}(2-n)\langle \nabla b^2, \nabla|\nabla b|^2\rangle \\
&= b^{-n}(2\ \Delta|\nabla b|^2\ b^2 + 2(2-n)\langle \nabla b^2, \nabla|\nabla b|^2\rangle)\ .
\end{aligned}
$$

□

In the following theorem we observe that the assumptions are applicable on any smooth manifold M^n, as it was also the case in the previous theorem .

Theorem 4.2.2 *Let (M^n, g) be an n-dimensional Riemannian manifold, with the assumption that $n \geq 3$. Then,*

$$
\left[r^{2-n}[A(r) - \mathrm{Vol}(\partial B_1(0))]\right]' = \frac{r^{1-n}}{2} \int\limits_{b\leq r} \left(\left|\mathrm{Hess}_{b^2} - \frac{\Delta b^2}{n} g\right|^2 + \mathrm{Ric}(\nabla b^2, \nabla b^2) \right) b^{-n}\, d\mathrm{Vol}\ .
$$

In particular, when Ricci curvature is non-negative, the function

$$
H(r) = r^{2-n}[A(r) - \mathrm{Vol}(\partial B_1(0))]
$$

is non-decreasing with respect to r.

Proof. We will first show a formula equivalent to the requested, one that is

$$
(2-n)[A(r) - \mathrm{Vol}(\partial B_1(0))] + rA'(r) = \frac{1}{2} \int\limits_{b\leq r} \left(\left|\mathrm{Hess}_{b^2} - \frac{\Delta b^2}{n} g\right|^2 + \mathrm{Ric}(\nabla b^2, \nabla b^2) \right) b^{-n}\, d\mathrm{Vol}\ . \quad (4.2.3)
$$

First we observe that for a special choise for the function v, the useful quantity $I_v(r)$ becomes

$$
I_{|\nabla b|^2 G}(r) = r^{1-n} \int\limits_{b=r} |\nabla b|^3 G\ dArea = r^{2-n} A(r)\ .
$$

For $r_2 > r_1 > 0$,

$$
\begin{aligned}
&r_2^{n-1}(r^{2-n}A)'(r_2) - r_1^{n-1}(r^{2-n}A)'(r_1) \\
=& r_2^{n-1} I'_{|\nabla b|^2 G}(r_2) - r_1^{n-1} I'_{|\nabla b|^2 G}(r_1) \\
=& r_2^{n-1} r_2^{1-n} \int\limits_{b=r_2} \left(|\nabla b|^2 G\right)_n dArea - r_1^{n-1} r_1^{1-n} \int\limits_{b=r_1} \left(|\nabla b|^2 G\right)_n dArea\ .
\end{aligned}
$$

Also we have from Stokes' theorem, that

$$
\int\limits_{b=r} \left(|\nabla b|^2 G\right)_n dArea = \int\limits_{b\leq r} \Delta(|\nabla b|^2 G) d\mathrm{Vol}
$$

and Lemma 4.2.1 now implies

$$\int_{b\le r_2} \Delta(|\nabla b|^2 G)d\mathrm{Vol} - \int_{b\le r_1} \Delta(|\nabla b|^2 G)d\mathrm{Vol}$$
$$=\frac{1}{2}\int_{r_1\le b\le r_2} \left(\left|\mathrm{Hess}_{b^2} - \frac{\Delta b^2}{n} g\right|^2 + \mathrm{Ric}(\nabla b^2, \nabla b^2)\right) b^{-n} d\mathrm{Vol}\ .$$

Moreover,

$$r^{n-1}(r^{2-n}A)'(r) = (2-n)A(r) + rA'(r)\ ,$$

as well as that there exists a sequence $r_i \to 0$ such that

$$(2-n)A(r_i) + r_i A'(r_i) \to (2-n)\mathrm{Vol}(\partial B_1(0))\ , \tag{4.2.4}$$

as $r_i \to 0$. This is true, because from Lemma 4.1.1, $A(r) \to \mathrm{Vol}(\partial B_1(0))$ as $r \to 0$ and this in turn implies that A is uniformly bounded for r sufficiently small. Hence there exists a sequence $r_i \to 0$ such that $r_i A'(r_i) \to 0$. Then from (4.2.4) we get that

$$(2-n)[A(r) - \mathrm{Vol}(\partial B_1(0))] + rA'(r) = \frac{1}{2}\int_{b\le r} \left(\left|\mathrm{Hess}_{b^2} - \frac{\Delta b^2}{n} g\right|^2 + \mathrm{Ric}(\nabla b^2, \nabla b^2)\right) b^{-n} d\mathrm{Vol}\ ,$$

which is equivalent to the original claim.

□

4.2.1 An alternative Second Monotonicity Formula

Now, we will mention an 'alternative' second monotonicity formula. We say alternative, because we will reformulate the Second Monotonicity Formula by defining a new 'volume' of balls on a Riemannian manifold, integrating again the appropriate function over the level sets of b. We do that by setting

$$V_\infty(r) = \int_{1\le b\le r} (|\nabla b|^2 - 1)|\nabla b|^2 b^{-n}\ d\mathrm{Vol}\ ,$$

and by the coarea formula

$$V_\infty(r) = \int_1^r s^{-n} \int_{b=s} (|\nabla b|^3 - |\nabla b|)\ dArea\ ds\ .$$

Hence, by the definition of $A(r)$ and Lemma 4.1.1 we have that its derivative satisfies

$$\frac{d}{dr}V_\infty = V_\infty'(r) = r^{-n}\int\limits_{b=r}(|\nabla b|^3 - |\nabla b|)\, dArea = \frac{A(r) - \mathrm{Vol}(\partial B_1(0))}{r} \ . \qquad (4.2.5)$$

We can now rewrite our second monotonicity theorem in terms of this second 'volume' of balls as follows.

Theorem 4.2.3 *Let (M^n, g) be a Riemannian manifold, with $n \geq 3$. Then the following holds:*

$$\big[A - (n-2)V_\infty\big]'(r) = \frac{1}{2r}\int\limits_{b\leq r}\left(\left|\mathrm{Hess}_{b^2} - \frac{\Delta b^2}{n}g\right|^2 + \mathrm{Ric}(\nabla b^2, \nabla b^2)\right) b^{-n} d\mathrm{Vol} \ .$$

Moreover, when M has non-negative Ricci curvature, the function

$$G(r) = A(r) - (n-2)V_\infty(r) \ ,$$

is non-decreasing with respect to r.

Proof. We have that

$$r\big[A - (n-2)V_\infty\big]'(r) = (2-n)[A(r) - \mathrm{Vol}(\partial B_1(0))] + rA'(r) \ ,$$

is equivalent to

$$(2-n)[A(r) - \mathrm{Vol}(\partial B_1(0))] = -r(n-2)\frac{A(r) - \mathrm{Vol}(\partial B_1(0))}{r} \ .$$

Therefore the desired result follows from (4.2.3).

□

Similarly to the situation after the first monotonicity formula, we get the following immediate corollary from this second monotonicity formula for manifolds with non-negative Ricci curvature.

Corollary 4.2.4 *If M is an n-dimensional Riemannian manifold with non-negative Ricci curvature, then for $r_2 > r_1 > 0$, we have that*

$$A(r_2) - (n-2)V_\infty(r_2) \geq A(r_2) - (n-1)V_\infty(r_1) \ .$$

Moreover, if in the above inequality, we have equality, then the set $\{x \in M : b(x) \leq r_2\}$ is isometric to a ball of radius r_2 in Euclidean space.

Proof. Since the function $G(r)$ is a non-decreasing function from the previous Theorem, it follows directly that

$$\text{for} \quad r_2 > r_1 \quad \Rightarrow \quad G(r_2) \geq G(r_1) \ ,$$

which is the claim.

□

Before the next theorem, we define the following function via V_∞,

$$J(s) = -(n-2)sV_\infty(s^{\frac{1}{(2-n)}}) \ .$$

Theorem 4.2.5 *Let M^n be an n-dimensional Riemannian manifold. Then for the function $J(s)$ holds the following properties*

i) $J'(s) = A(s) - \mathrm{Vol}(\partial B_1(0)) - (n-2)V_\infty(s^{\frac{1}{(2-n)}})$,

ii) $J''(s) = -\dfrac{1}{2(n-2)s} \displaystyle\int_{b \leq s^{\frac{1}{(2-n)}}} \left(\left|\mathrm{Hess}_{b^2} - \frac{\Delta b^2}{n} g\right|^2 + \mathrm{Ric}(\nabla b^2, \nabla b^2) \right) b^{-n} d\mathrm{Vol}$.

Moreover, when the Ricci curvature is non-negative, $J''(s) \leq 0$ and thus the function $J'(s)$ is non-increasing with respect to s.

Proof. For the first claim we have that

$$\begin{aligned} J'(s) &= -(n-2)V_\infty(s^{\frac{1}{2-n}}) + s^{\frac{1-n}{2-n}} \int_{b = s^{\frac{1}{(2-n)}}} \left(|\nabla b|^3 - |\nabla b|\right) dArea \\ &= -(n-2)V_\infty(s^{\frac{1}{2-n}}) + A(s^{\frac{1-n}{2-n}}) - \mathrm{Vol}(\partial B_1(0)) \ . \end{aligned}$$

Now by setting the integral equal to I, over the level set $b = s^{\frac{1}{2-n}}$ in this time, Theorem 4.2.3 gives,

$$J''(s) = \left[A - (n-2)V_\infty\right]'(s^{\frac{1}{2-n}}) = -\frac{1}{2(n-2)} s^{-1} I \ .$$

which is (ii).

□

4.2.2 Asymptotic Description of $A(r)$ and $V(r)$

The next theorem give us information about the behavior of $A(r)$ and $V(r)$, as r tends to infinity. It is also in substance, the first result that shows us how the geometry of a manifold affects the asymptotic behavior of Green's function at infinity.

Note that this result is in the opposite direction of Lemma 4.1.1, mentioned at the beginning of this Chapter. We use a paper of Colding-Minicozzi ([7]) to calculate this asymptotic description of $A(r)$ and $V(r)$ for manifolds with non-negative Ricci curvature.

At first, we have from the Bishop-Gromov theorem that the function

$$f(r) = \frac{\text{Vol}(B_x(r))}{\text{Vol}(B_x^k(r))}$$

is bounded ($0 \leq f(r) \leq 1$) and also decreasing in the radius r and hence the quotient $\frac{\text{Vol}(B_x(r))}{r^n}$ converges either at 0, either at infinity. The last case is the interesting one and thus we have the following definition.

Definition 4.2.6 *On an n-dimensional Riemannian manifold (M^n, g) we define*[3] *the*

$$V_M = \lim_{r\to\infty} \frac{\text{Vol}B_x(r)}{r^n} \ ,$$

where $\text{Vol}(B_x(r))$ *is the volume of the geodesic ball centered at $x \in M$ of radius r.*

This quantity is relevant, because as we know from the Bishop-Gromov theorem, the volumes of the geodesic balls increase mostly like the Euclidean volumes of balls. As such, we distinguish two cases:

When our manifold M has non-negative[4] Ricci curvature, then we say that:

- M has Euclidean volume growth, if $V_M > 0$ which means that the geodesic volume rises like Euclidean (r^n)

- M has sub-Euclidean volume growth, if $V_M = 0$ which in turn means that the geodesic volume rises with less speed than the Euclidean

Theorem 4.2.7 *If (M^n, g) is a Riemannian manifold with non-negative Ricci curvature, then*

$$\lim_{r\to\infty} \frac{A(r)}{\text{Vol}(\partial B_1(0))} = \left(\frac{V_M}{\text{Vol}(B_1(0))}\right)^{\frac{2}{n-2}} \ , \qquad (4.2.6)$$

$$\lim_{r\to\infty} \frac{V(r)}{\text{Vol}(B_1(0))} = \left(\frac{V_M}{\text{Vol}(B_1(0))}\right)^{\frac{2}{n-2}} \ . \qquad (4.2.7)$$

Proof. We first observe that in both the left sides of the two claims above, the asymptotic behavior at infinity of $A(r)$ and $V(r)$, is directly related to V_M on the

[3] You can also see ([6], p 14).

[4] Note that for the Hyperbolic space, we have $V_M = \infty$.

right sides of these two equalities. By the Bishop-Gromov volume comparison theorem, if $r \geq r_0 \geq 0$, then

$$\frac{\mathrm{Vol}(B_x(r))}{\mathrm{Vol}(B_x^k(r))} \leq \frac{\mathrm{Vol}(B_x(r_0))}{\mathrm{Vol}(B_x^k(r_0))} \ .$$

Now, when $k = 0$, and then $M_k^n = \mathbb{R}^n$, we get that

$$\frac{\mathrm{Vol}(B_x(r))}{r^n \mathrm{Vol}(B_1(0))} \leq \frac{\mathrm{Vol}(B_x(r_0))}{r_0^n \mathrm{Vol}(B_1(0))} \ ,$$

which implies

$$r^{-n}\mathrm{Vol}(B_x(r)) \leq r_0^{-n}\mathrm{Vol}(B_x(r_0)) \ .$$

Then, by the Li-Yau ([9]) lower bound for the Green function,

$$C \int_{d(x,y)}^{\infty} \frac{s}{\mathrm{Vol}(B_x(s))} ds \leq G(x,y) \ . \tag{4.2.8}$$

More precisely, by Theorem 5.2 in [9], we have that there exist constants a, b depending only on n, such that

$$a \int_{r^2}^{\infty} V^{-1}(B_x(\sqrt{t}))dt \leq G(x,y) \leq b \int_{r^2}^{\infty} V^{-1}(B_x(\sqrt{t}))dt \ ,$$

where $r(x,y)$ is the distance function and $B_x(\sqrt{t})$ is a geodesic ball of radius $\sqrt{t}$, centered at some $x \in M$.

Therefore, straightforward computations and the Bishop-Gromov inequality, give (4.2.8). It follows that if $d(x,y) \geq 0$ then,

$$G(x,y) \geq \frac{r_0^n \ C}{\mathrm{Vol}(B_x(r_0))} d^{2-n}(x,y) \ .$$

Indeed, this happens since

$$G(x,y) \geq \ C \int_{d(x,y)}^{\infty} \frac{s}{\mathrm{Vol}(B_x(s))} ds \geq \ C \int_{d(x,y)}^{\infty} \frac{s \ r_0^n}{s^n \ \mathrm{Vol}(B_x(r_0))} ds \ ,$$

and the last is equal to

$$\frac{C \ r_0^n}{\mathrm{Vol}(B_x(r_0))} \lim_{a \to \infty} \int_{d(x,y)}^{a} s^{1-n} ds \ .$$

Note that this lower bound for the Green function is global on M.

Now by the Cheng-Yau [4] gradient estimate[5], we have that on a complete Riemannian manifold M^n, $n \geq 2$, with $Ric \geq -\mathcal{K}$ and $\mathcal{K} \geq 0$, if u is a positive harmonic function in a geodesic ball $B_{x_0}(r) \subset M$, then

$$\frac{|\nabla u|}{u} \leq \frac{c_n}{r} + c_n \sqrt{\mathcal{K}}\ ,$$

holds in $B_{x_o}(\frac{r}{2})$, where c_n depends only on the dimension n.

This estimate applied to the harmonic function G, and setting $\mathcal{K} = 0$, implies that at such a y, into the ball $B_y(\frac{r}{2})$, with $\frac{r}{2} < d(x, y)$ holds,

$$\begin{aligned} \frac{|\nabla G(y)|}{G(y)} \leq \frac{c_n}{r} \Rightarrow \frac{|\nabla b(y)^{2-n}|}{b(y)^{2-n}} \leq \frac{c_n}{r} \\ \Rightarrow (2-n)|\nabla \log b(y)| \leq \frac{c_n}{r}\ , \end{aligned}$$

and hence follows

$$|\nabla b| \leq \frac{Cb}{r}\ ,$$

for some constant $C = C(n)$.

This, together with the fact that

$$\frac{Cb}{r} = C\frac{G^{\frac{1}{2-n}}}{r} \leq C\left[r_0^{-n}\ \mathrm{Vol}(B_{r_0}(x))\right]^{\frac{1}{n-2}}\ ,$$

implies

$$|\nabla b| = b|\nabla \log b| \leq C\frac{G^{\frac{1}{2-n}}}{r} \leq C\left[r_0^{-n}\ \mathrm{Vol}(B_{r_0}(x))\right]^{\frac{1}{n-2}}\ . \tag{4.2.9}$$

At this point, we will distinguish the following cases: First when M has Euclidean volume growth and second, when M has a sub-Euclidean volume growth.

Then, if M has sub-Euclidean volume growth, i.e. $V_M = 0$, (4.2.9) implies that there exists r_0 such that $|\nabla b| \leq C\big(\frac{B_x(r_0)}{r_0^n}\big)^{\frac{1}{n-2}}$, $\forall r \geq r_0$ .This assumption together with the fact that $\lim_{r(x)\to\infty} |\nabla b(x)| = 0$ implies $\lim_{r(x)\to\infty} \sup_{b(x)\geq r} |\nabla b(x)| = 0$, where it is assumed from the minimality of Green's function that, $b(x) \geq r \neq \varnothing$, implies

$$\begin{aligned} A(r) &= r^{1-n} \int\limits_{b=r} |\nabla b|^3\ dArea \\ &\leq r^{1-n} \int\limits_{b=r} |\nabla b| \left[\sup_{b=r} |\nabla b|^2\right]\ dArea \end{aligned}$$

[5]You can also see ([18], 4.6).

$$\leq \left(r^{1-n} \int\limits_{b=r} |\nabla b| \; dArea \right) \sup_{b \geq r} |\nabla b|^2$$
$$= I_1(r) \; \sup_{b \geq r} |\nabla b|^2 \to 0 \; ,$$

as $r \to \infty$. From this, (4.2.6) follows directly since,

$$\lim_{r \to \infty} \frac{A(r)}{\mathrm{Vol}(\partial B_1(0))} = 0 \; .$$

The second claim also holds, by a similar argument. Indeed, since

$$V(r) = r^{-n} \int\limits_{b \leq r} |\nabla b|^4 \; d\mathrm{Vol},$$

and by letting an arbitary $r_0 > 0$, we have that

$$\begin{aligned} V(r) &= r^{-n} \left(\int\limits_{b \leq r_0} |\nabla b|^4 \; d\mathrm{Vol} + \int\limits_{r_0 \leq b \leq r} |\nabla b|^4 \; d\mathrm{Vol} \right) \\ &\leq r^{-n} \int\limits_{b \leq r_0} |\nabla b|^4 \; d\mathrm{Vol} + \left[\sup_{b \geq r_0} |\nabla b|^2 \right] r^{-n} \int\limits_{b \leq r} |\nabla b|^2 \; d\mathrm{Vol} \\ &= \frac{h(r_0)}{r^n} + \frac{I_1(r)}{n} \; \sup_{b \geq r_0} |\nabla b|^2 \; . \end{aligned}$$

Since $h(r_0) := \int\limits_{b \leq r_0} |\nabla b|^4 \; d\mathrm{Vol}$ does not depend on r, we have

$$\lim_{r \to \infty} V(r) \leq 0 + \frac{I_1(r)}{n} \; \sup_{b \geq r_0} |\nabla b|^2 \; ,$$

and by taking the limit as $r_0 \to \infty$ on the both sides of the last inequality, we get

$$\lim_{r \to \infty} V(r) \leq \left[\lim_{r_0 \to \infty} \sup_{b \geq r_0} |\nabla b|^2 \right] \frac{\mathrm{Vol}(\partial B_0(1)}{n} = 0 \; .$$

Now, we suppose that M has Euclidean volume growth. We set $\omega = \mathrm{Vol}(B_1(0))$ and $\lambda = \left[\frac{V_M}{\omega} \right]^{\frac{1}{n-2}}$. Then from (3.37) and (3.38) in [7], we have that $\forall \delta > 0$, $\exists R_\delta$ such that, for $r > R = R_\delta$,

$$i) |\lambda^{-1} b - r| < \delta r \; ,$$
$$ii) \int_{b \leq r} \left| \lambda^{-2} |\nabla b|^2 - 1 \right|^2 \leq \delta \; \mathrm{Vol}\left(\{ x : b(x) \leq r \} \right) \; ,$$

with $\lambda^{-1}b = r(1+o(1))$, when $r \to 0$.

We wish to show that

$$|\lambda^{-2}V(r) - \omega| \to 0 \ ,$$

as r tends to infinity. Indeed, for $r > R$ the Cauchy-Schwarz inequality together with (ii) implies

$$\begin{aligned}
|\lambda^{-2}V(r) - \omega| &= \Big| r^{-n} \int_{b \le r} \lambda^{-2} |\nabla b|^4 d\mathrm{Vol} - r^{-n} \int_{b\le r} |\nabla b|^2 dVol \Big| \\
&\le r^{-n} \int_{b\le r} \big| \lambda^{-2}|\nabla b|^2 - 1 \big| |\nabla b|^2 d\mathrm{Vol} \\
&\le \left[r^{-n} \int_{b\le r} \big| \lambda^{-2}|\nabla b|^2 - 1 \big| \right]^{\frac{1}{2}} V(r)^{\frac{1}{2}} \\
&\le \delta \left[\frac{\mathrm{Vol}\left(\{x : b(x) \le r\}\right)}{r^n} \right]^{\frac{1}{2}} V(r)^{\frac{1}{2}} \\
&\le C\delta \ .
\end{aligned}$$

Note that in the last inequality used the fact that $\left[\frac{\mathrm{Vol}(\{x:b(x)\le r\})}{r^n}\right]$ is bounded. This argument follows from (i) and the Bishop-Gromov inequality, since

$$\big| \lambda^{-1} b - r \big| < \delta r$$

implies that for t large,

$$\{b \le t\} \subseteq \{r \le \frac{t}{\lambda(1-\delta)}\} \ ,$$

and therefore

$$\begin{aligned}
\left(t^{-n}\mathrm{Vol}(\{b \le t\})\right)^{\frac{1}{2}} &\le \left(t^{-n} \mathrm{Vol}(\{r \le \frac{t}{\lambda(1-\delta)}\}) \right)^{\frac{1}{2}} \\
&\le \left[t^{-n} \mathrm{Vol}(B_0(1)) \left(\frac{t}{\lambda(1-\delta)} \right)^n \right]^{\frac{1}{2}} \\
&= \left(\mathrm{Vol}(B_0(1))\right)^{\frac{1}{2}} \left[\lambda(1-\delta)\right]^{\frac{-n}{2}} \ .
\end{aligned}$$

This, combined with the fact that $|\nabla b|$ is also bounded by (4.2.9), implies that $V(r)$ is also bounded.

Consequently, we have (4.2.7) since

$$|\lambda^{-2}V(r) - \omega| \to 0 \ ,$$

as $r \to \infty$, or equivalently

$$\lim_{r\to\infty} \frac{V_M}{\omega} = \lambda^2 = \left(\frac{V_M}{\omega}\right)^{\frac{2}{n-2}} \ . \tag{4.2.10}$$

Now for (4.2.6), we consider $r_0 > 0$ such that

$$\omega\lambda^2 - \epsilon \leq V(r) \leq \omega\lambda^2 + \epsilon \ ,$$

for $r > r_0$. This argument comes from (4.2.10).

We will use the following expession for $V(r)$:

$$V(r) = r_{-n} \int_0^r s^{n-1} A(s) \ ds \ .$$

Also we need the first monotonicity formula, which says that

$$[A(r) - 2(n-1)V(r)]' \geq 0 \ ,$$

with the assumption for the Ricci curvature to be non-negative.

Thus, for $r > r_0$, we have from Corollary 4.1.5 of first monotonicity Theorem, that

$$A(r) \geq A(r_0) + 2(n-1)\left[V(r) - V(r_0)\right] \geq A(r_0) - 4(n-1)\epsilon \ , \tag{4.2.11}$$

and then

$$V(r) \geq r^{-n} \left[\int_0^{r_0} s^{n-1} A(s) \ ds + \int_{r_0}^{r} s^{n-1} \left(A(r_0) - 4(n-1)\epsilon\right) \ ds \right]$$

$$= \frac{C}{r^n} + \frac{A(r_0) - 4(n-1)\epsilon}{n} \longrightarrow \frac{A(r_0) - 4(n-1)\epsilon}{n} \ ,$$

as $r \to \infty$.

This implies that for $r_0 \to \infty$ and $\epsilon \to 0$,

$$\omega\lambda^2 = \lim_{r\to\infty} V(r) \geq \frac{A(r_0) - 4(n-1)\epsilon}{n} \ ,$$

and it also follows that

$$\limsup_{r\to\infty} A(r) \leq \frac{\omega\lambda^2}{n} \ .$$

From (4.2.7) and the first monotonicity formula, the limit of $A(r)$ at infinity exists.

Finally, we have again by the second expression of $V(r)$, that

$$\lim_{r\to\infty} A(r) = n \lim_{r\to\infty} V(r) \ ,$$

and

$$\lim_{r\to\infty} \frac{A(r)}{\alpha_{n-1}} = \frac{n\omega_n}{\alpha_{n-1}}\lambda^2 = \lambda^2 \ ,$$

which is (4.2.6). □

From the above Theorem and (4.2.5), the following characterization of Euclidean space as the only manifold with non-negative Ricci curvature where V_∞ is bounded follows directly.

Corollary 4.2.8 *Let M^n be a manifold with non-negative Ricci curvature. Then*

$$\inf V_\infty > -\infty \ ,$$

if and only if M is Euclidean space.

Proof. The opposite direction of this equivalence is direct, because when M is the Euclidean space, we obviously have that $V_\infty = 0$ and hence

$$\inf V_\infty > -\infty \ .$$

Now, for the right direction, we suppose that M is not the Euclidean space and we expect to show that $\inf V_\infty = -\infty$.

For this purpose, we will use a classical argument from Comparison Geometry :

Lemma 4.2.9 *If for the Ricci curvature of a manifold M holds $Ric \geq 0$, then*

$$\frac{V_M}{\mathrm{Vol}(B_1(0))} \leq 1 \ ,$$

with the equality holds if and only if M is the Euclidean space.

Proof. Since M is not the Euclidean space, Bishop-Gromov theorem allows us to choose an arbitrary r_0 such that

$$\frac{\mathrm{Vol}(B_x(r_0))}{\mathrm{Vol}(B_x^k(r_0))} < 1 \ , \qquad \forall r > r_0 \ .$$

Note that when $k = 0$, i.e. we are in the Euclidean space, $B_x^k(r_0) = B_0(r_0)$ is a ball with radius r_0. Thus we have that

$$\frac{\mathrm{Vol}(B_x(r))}{\mathrm{Vol}(B_x^k(r))} \leq \frac{\mathrm{Vol}(B_x(r_0)}{\mathrm{Vol}(B_x^k(r_0))} := c < 1$$

and

$$\frac{V_M}{\mathrm{Vol}(B_1(0)} = \lim_{r\to\infty} \frac{\mathrm{Vol}(B_x(r))}{\mathrm{Vol}(B_0(r))} \leq c < 1 \ ,$$

when $k = 0$.

□

Now, from (4.2.6) we have,

$$\lim_{r\to\infty} \frac{A(r)}{\mathrm{Vol}(\partial B_1(0))} = \left[\frac{V_M}{\mathrm{Vol}(B_1(0))}\right]^{\frac{2}{n-2}} \leq c^{\frac{2}{n-2}} < 1 \ .$$

Therefore, there exists an $r_1 > 0$ such that $\frac{A(r)}{\mathrm{Vol}(\partial B_1(0))} < \frac{c^{\frac{2}{n-2}}+1}{2} = c' < 1$, for each $r > r_1$.

Now we have,

$$\begin{aligned}
V_\infty(r) &= V_\infty(r_1) + \int_{r_1}^{r} V_\infty'(t)\ dt \\
&= V_\infty(r_1) + \mathrm{Vol}(\partial B_1(0)) \int_{r_1}^{r} \frac{\frac{A(t)}{\mathrm{Vol}(\partial B_1(0))} - 1}{t}\ dt \\
&\leq V_\infty(r_1) + \mathrm{Vol}(\partial B_1(0)) \int_{r_1}^{r} \frac{c'-1}{t}\ dt \\
&= V_\infty(r_1) + \mathrm{Vol}(\partial B_1(0))(c'-1)(\log r - \log r_1) \ .
\end{aligned}$$

Letting $r \to \infty$, we have the result.

□

4.3 The $\mathcal{L}$ operator and estimates for b

Define a drift Laplacian on the manifold M by

$$\begin{aligned}
\mathcal{L}u &= \frac{1}{G^2}\ \mathrm{div}(G^2\nabla u) \\
&= \Delta u + G^{-2}\nabla G^2 \nabla u \\
&= \Delta u + 2\langle \nabla \log G, \nabla u\rangle \ .
\end{aligned}$$

The next Lemma is useful for proving results which apply to 3- as well as to 4-manifolds. To prove this, we will need Lemma 4.2.1 together with the $\mathcal{L}$ operator.

Lemma 4.3.1 *Let M^n be a Riemannian manifold with non-negative Ricci curvature and G be a positive harmonic function defined as before by $b^{2-n} = G$. Then we have*

$$\begin{aligned}
&\text{i)}\ \ \mathcal{L}|\nabla b|^2 = \frac{1}{2b^2}\left(\left|Hess_{b^2} - \frac{\Delta b^2}{n} g\right|^2 + \mathrm{Ric}(\nabla b^2, \nabla b^2)\right) \ , \\
&\text{ii)}\ \ \mathcal{L}b^2 = 2(4-n)|\nabla b|^2 \ , \\
&\text{iii)}\ \ \mathcal{L}b^{n-2} = 0 \ .
\end{aligned}$$

Proof. Since

$$\mathcal{L}|\nabla b|^2 = \Delta\big(|\nabla b|^2\big) + 2\langle \nabla \log G, \nabla |\nabla b|^2\rangle \ ,$$

we have from Lemma 4.2.1 that the first claim holds if and only if

$$\begin{aligned} 2\langle \nabla \log G, \nabla |\nabla b|^2\rangle &= b^{-2}(2-n)\langle \nabla b^2, \nabla |\nabla b|^2\rangle \\ \Leftrightarrow b^{-2}\langle G^{\frac{n}{2-n}} G^{-1}\nabla G, \nabla |\nabla b|^2\rangle &= \langle \nabla \log G, \nabla |\nabla b|^2\rangle \ , \end{aligned}$$

which obviously holds.

The second claim follows from straightforward computations, using the fact that $\Delta b^2 = 2n|\nabla b|^2$. Finally, since $\nabla b^{n-2} = \nabla G^{-1} = -G^{-2}\nabla G$ and G is harmonic, we have

$$\begin{aligned} \mathcal{L} b^{n-2} &= G^{-2}\mathrm{div}\big(G^2 \nabla b^{n-2}\big) \\ &= G^{-2}\mathrm{div}\big[G^2(-G^{-2}\nabla G)\big] \\ &= -G^{-2}\Delta G = 0 \ . \end{aligned}$$

□

From this Lemma it follows that on a manifold with non-negative Ricci curvature $Hess_{b^2}$ is a multiple of the identity, at a maximum for $|\nabla b|^2$. This is because, since a smooth function u has a maximum at $p \in M$, it follows that $\nabla u(p) = 0$, $\Delta u(p) \leq 0$ as well as $\mathcal{L}u = \Delta u + 2\langle \nabla \log G, \nabla u\rangle = \Delta u$. Now for $u = |\nabla b|^2$ and since the Ricci curvature is non-negative, we have that

$$0 \geq \Delta\big(|\nabla b|^2\big) = \mathcal{L}|\nabla b|^2 = \frac{1}{2b^2}\left(\big|\mathrm{Hess}_{b^2} - \frac{\Delta b^2}{n} g\big|^2 + \mathrm{Ric}(\nabla b^2, \nabla b^2)\right) \geq 0 \ .$$

This implies immediately that

$$\mathrm{Hess}_{b^2} = \frac{\Delta b^2}{n} g \ , \tag{4.3.1}$$

and

$$\mathrm{Ric}(\nabla b^2, \nabla b^2) = 0 \ . \tag{4.3.2}$$

Then (4.3.1) in turn implies that at a maximum point of $|\nabla b|^2$ we have

$$\mathrm{Hess}_{b^2} = \frac{2n|\nabla b|^2}{n} g = 2|\nabla b|^2 g \ .$$

The first two inequalities of the Lemma are proven assuming that G is the Green function, whereas the third inequality holds for any positive harmonic function G, with $\frac{1}{G}$ proper.

Lemma 4.3.2 *On a manifold with non-negative Ricci curvature, if $x \in M$ is a fixed point and r is the distance to x, then we have*

$$\begin{aligned} &\text{i)} \quad b \leq r \ , \\ &\text{ii)} \quad |\nabla b| \leq C = C(n) \ , \\ &\text{iii)} \quad \mathcal{L}|\nabla b|^2 \geq 0 \ . \end{aligned}$$

Proof. The last claim was used as an intermediate argument in the proof of a previous Lemma and therefore it applies here since we have the same conditions for Ricci curvature.

To prove the other two claims, observe first of all that by the Laplace comparison theorem[6], we have

$$\begin{aligned} \Delta r^{2-n} &= (2-n)r^{1-n}\Delta r + (2-n)(1-n)r^{-n}|\nabla r|^2 \\ &\geq -(n-2)r^{1-n}(n-1)r^{-1} + (n-2)(n-1)r^{-n} \\ &= 0 \ , \end{aligned}$$

and then since $f(0) = 0$, with $f := r^{2-n} - G$ and $\limsup_{r\to\infty} f \leq 0$, we have

$$r^{2-n} - G \leq 0 \ ,$$

by the maximum principle. Therefore, on such a manifold one has

$$b \leq r \ ,$$

which is the first claim.

For the second claim, we have from an intermediate result of Theorem 4.2.7, based in the Cheng-Yau gradient estimate, that

$$b|\nabla \log b| \leq C \ G^{\frac{1}{2-n}} \ ,$$

for some constant C, which depends only on the dimension of the manifold. This implies that

$$|\nabla b| \leq \frac{b\ C(n)}{r} \leq C(n) \tag{4.3.3}$$

and from (i) we have the result.

□

The following Lemma connects the derivative of $I_u(r)$, where u is a smooth function, with the $\mathcal{L}$ operator applied on u. The result of this Lemma will be used to study the monotonicity of $I_u(r)$, when u is bounded from above.

[6]For the Laplacian comparison theorem see Appendix 6.3.

Lemma 4.3.3 *Let M^n be a manifold and suppose that $u : M \to \mathbb{R}$ is a smooth function. Then for $r_2 > r_1 > 0$,*

$$I'_u(r_2) = r_2^{n-3} r_1^{3-n} I'_u(r_1) + r_2^{n-3} \int\limits_{r_1 \le b \le r_2} G^2 \mathcal{L}u \, d\mathrm{Vol} \ . \tag{4.3.4}$$

Proof. Observe first that,

$$\begin{aligned} r_1^{3-n} I'_u(r_1) &= r_1^{3-n} r_1^{1-n} \int\limits_{b=r_1} u_n \, dArea \\ \Rightarrow r_2^{n-3} r_1^{3-n} I'_u(r_1) &= r_2^{n-3} r_1^{4-2n} \int\limits_{b=r_1} u_n \, dArea \ . \end{aligned}$$

Now for $r_2 > r_1 > 0$, we will get the desired result by calculating the difference

$$I'_u(r_2) - r_2^{n-3} \int\limits_{r_1 \le b \le r_2} G^2 \mathcal{L}u \, d\mathrm{Vol} \ .$$

So we have

$$\begin{aligned} & r_2^{1-n} \int\limits_{b \le r_2} \Delta u \, d\mathrm{Vol} - r_2^{n-3} \int\limits_{r_1 \le b \le r_2} G^2 \mathcal{L}u \, d\mathrm{Vol} \\ = \ & r_2^{1-n} \int\limits_{b=r_2} u_n \, dArea - r_2^{n-3} \int\limits_{r_1 \le b \le r_2} \mathrm{div}\big(G^2 \nabla u\big) \, d\mathrm{Vol} \\ = \ & r_2^{1-n} \int\limits_{b=r_2} u_n \, dArea - r_2^{n-3} \left[\int\limits_{b=r_2} G^2 u_n \, dArea - \int\limits_{b=r_1} G^2 u_n \, dArea \right] \\ = \ & r_2^{n-3} r_1^{4-2n} \int\limits_{b=r_1} u_n \, dArea \ . \end{aligned}$$

□

Corollary 4.3.4 *Let M^n be a manifold with $n \ge 3$ and suppose that u is an $\mathcal{L}$-subharmonic function that is bounded from above. Then*

$$I_u(r) = r^{1-n} \int\limits_{b=r} u |\nabla b| \, dArea$$

is non-increasing.

Proof. Since u is an $\mathcal{L}$-subharmonic function, we have that $\mathcal{L}u \geq 0$. Then it follows from Lemma 4.3.3 that for $r_2 > r_1 > 0$,

$$I_u'(r_2) \geq r_2^{n-3} r_1^{3-n} I_u'(r_1) \ .$$

As u is bounded from above, I_u is also bounded from above. Now, recall that as $r \to 0$,

$$r^{-n} \int\limits_{b \leq r} 1 \ d\mathrm{Vol} \to \mathrm{Vol}(B_1(0)) \ ,$$

as well as

$$|\Delta u| \leq C \ .$$

Hence, by the triangle inequality, we have

$$|I_u'| \leq r^{1-n} \int\limits_{b \leq r} |\Delta u| \to 0 \ , \tag{4.3.5}$$

as $r \to 0$.

Eventually we conclude that the function $I_u(r)$ is non-increasing with respect to r, i.e.

$$I_u'(r) \leq 0 \ .$$

To see this, suppose $I_u'(r_0) > 0$ for some $r_0 \in \mathbb{R}$ and integrate $I_u(r) \geq r^{n-3} r_0^{3-n} I_u'(r_0)$, on $[r_0, \infty)$. This gives a contradiction, since by hypothesis u and therefore I_u is bounded.

□

The next Corollary gives us the expression of $I_u'(r)$ on 3 and 4-manifolds.

Remark: For a smooth function $u : M \to \mathbb{R}$ we have that

$$i) \quad \lim_{r \to 0} I_u'(r) = 0 \quad \text{and}$$

$$ii) \quad \lim_{r \to 0} \frac{I_u'(r)}{r} = \mathrm{Vol}(B_1(0)) \ \Delta u(x) \ .$$

For the second claim, we have

$$\lim_{r \to 0} \frac{I_u'(r)}{r} = \lim_{r \to 0} r^{-n} \int\limits_{b \leq r} \Delta_u \ d\mathrm{Vol} = \mathrm{Vol}(B_1(0)) \ \Delta_u(x) \ ,$$

where the last equality holds from Lemma 4.1.1.

Corollary 4.3.5 *Let M^n be an n-dimensional manifold and suppose that the function $u : M \to \mathbb{R}$, is smooth. Then*

i) *if* $n = 3, \ I_u'(r) = \int\limits_{b \leq r} G^2 \ \mathcal{L}u \ d\mathrm{Vol}$ *and*

$$\text{ii)} \quad \textit{if} \;\; n = 4, \;\; I'_u(r) = r \; \text{Vol}(\partial B_1(0)) \; \Delta_u(x) + r \int_{b \le r} G^2 \; \mathcal{L}u \; d\text{Vol} \; .$$

Proof. The proof follows directly by the Lemma 4.3.3, together with the above remark, both for cases $n = 3$ and $n = 4$.

□

4.4 Third Monotonicity formula

Corollary 4.4.1 *On any Riemannian manifold (M^n, g) with non-negative Ricci curvature and $n \geq 3$, we have that $A(r)$, $V(r)$ and $V_\infty(r)$ are non-increasing functions with respect to r and bounded from above by the same bounds as on $\mathbb{R}^n$. We have also that $A(r) \leq nV(r)$.*

Proof. By Lemma 4.3.2 we have that $|\nabla b|^2$ is an $\mathcal{L}$-subharmonic function, i.e. $\mathcal{L}|\nabla b|^2 \geq 0$. Hence by Corollary 4.3.4, since $I_u(r)$ is non-increasing function in r, we have that

$$A(r) = I_{|\nabla b|^2}(r) \; ,$$

is also non-increasing. Moreover, since A starts off at what it is in Euclidean space by Lemma 4.1.1, we get the first claim, that is the *generalized normalized* Area $A(r)$ is a non-increasing function and is bounden from above by the constant $\text{Vol}(\partial B_1(0))$.

Now, since the function $F(r)$ in the First Monotonicity formula, is non-decreasing (with our assumptions), we have that

$$A'(r) \geq 2(n-1)V'(r) \; .$$

This, together with the fact that $A'(r) \leq 0 \; \forall r$, proves the claim also for $V(r)$, since it starts being equal to what it is in Euclidean space, by Lemma 4.1.1.

Finally, the Second Monotonicity formula gives us that when Ricci curvature is non-negative, then the $U(r)$ function is non-decreasing, i.e.

$$A'(r) \geq (n-2)V'_\infty(r) \; .$$

Hence $V_\infty(r)$ is non-increasing in r. For the upper bound of this second volume of balls, we have from the second claim, together with Lemma 4.1.2 that

$$\frac{1}{r}\left[A(r) - n \; V(r)\right] \leq 0 \; .$$

□

For the next Theorem, which is the Third Monotonicity formula, we will need Lemmas 4.3.1 and 4.3.3.

Theorem 4.4.2 *Let (M^n, g) be an n-dimensional Riemannian manifold. Then for $r_2 > r_1 > 0$ we have*

$$r_2^{3-n}A'(r_2) - r_1^{3-n}A'(r_1) = \frac{1}{2}\int\limits_{r_1 \le b \le r_2}\left(\left|\mathrm{Hess}_{b^2} - \frac{\Delta b^2}{n}g\right|^2 + \mathrm{Ric}(\nabla b^2, \nabla b^2)\right) b^{2-2n}\, d\mathrm{Vol}.$$

This implies that, when the Ricci curvature on a M^n is non-negative, then the function

$$U(r) := r^{3-n}A'(r) \ ,$$

will be non-decreasing in r.

Proof. For $r_2 > r_1 > 0$ and by multiplying (4.3.4) in Lemma 4.3.3 with r_2^{3-n}, we get

$$r_2^{3-n}I'_u(r_2) = r_1^{3-n}I'_u(r_1) + \int\limits_{r_1 \le b \le r_2} G^2\mathcal{L}u\ d\mathrm{Vol}\ . \tag{4.4.1}$$

Now, since

$$I'_{|\nabla b|^2}(r) = A'(r)\ ,$$

it follows from (i) of Lemma 4.3.1, that

$$\begin{aligned}\int\limits_{r_1 \le b \le r_2} G^2\mathcal{L}|\nabla b|^2\ dVol &= \frac{1}{2}\int\limits_{r_1 \le b \le r_2}\left(\left|\mathrm{Hess}_{b^2} - \frac{\Delta b^2}{n}g\right|^2 + \mathrm{Ric}(\nabla b^2, \nabla b^2)\right)\frac{G^2}{b^2}\ d\mathrm{Vol} \\ &= \frac{1}{2}\int\limits_{r_1 \le b \le r_2}\left(\left|\mathrm{Hess}_{b^2} - \frac{\Delta b^2}{n}g\right|^2 + \mathrm{Ric}(\nabla b^2, \nabla b^2)\right) b^{2-2n}\ d\mathrm{Vol}.\end{aligned}$$

Then the claim follows directly from (4.4.1).

□

The first consequence of the Third monotonicity formula is the following Corollary, in which we will present a different expression of $A(r)$ (compare to the one we had at first monotonicity formula; here the integration occurs over the level sets $\{b \ge r\}$). More generally, the assymptotic description of $A(r)$ and $V(r)$, is the one that urged us to study what happens in supplements of the level sets $\{b \le r\}$.

Corollary 4.4.3 *If M^n is a Riemannian manifold with non-negative Ricci curvature, then*

$$A'(r) = -\frac{r^{n-3}}{2}\int\limits_{b \ge r}\left(\left|\mathrm{Hess}_{b^2} - \frac{\Delta b^2}{n}g\right|^2 + \mathrm{Ric}(\nabla b^2, \nabla b^2)\right) b^{2-2n}\ d\mathrm{Vol}\ .$$

Proof. Firstly we note that since G is minimal, we have $\{b \ge r\} \neq \varnothing$.

By letting $r_2 \to \infty$ in the above Theorem, we easily have that $r_2^{3-n}A'(r_2) = 0$. Hence the claim follows, for every $r > 0$.

□

Corollary 4.4.4 *Let (M^3, g) be a 3-dimensional Riemannian manifold. If $|\nabla b|^2$ is $\mathcal{C}^2$ function in a neighborhood of $x \in M$, then*

$$A'(r) = -\frac{1}{2} \int\limits_{b \leq r} \left(\left|\mathrm{Hess}_{b^2} - \frac{\Delta b^2}{n} g\right|^2 + \mathrm{Ric}(\nabla b^2, \nabla b^2) \right) b^{-4} \, d\mathrm{Vol} \ .$$

In addition, if M has non-negative Ricci curvature, then M is flat $\mathbb{R}^3$.

Proof. The first claim follows easily from Lemma 4.3.1, Corollary 4.3.4 and (i) of Corollary 4.3.5.

To prove the second claim, note first that $A'(r) = 0$ since $A(r)$ is non-increasing function in r. Hence $A(r)$ is constant from Corollary 4.4.1, together with the first claim. From this, as we have already seen, it follows directly that

$$\mathrm{Hess}_{b^2} = \frac{\Delta b^2}{n} g \qquad \text{and} \qquad \mathrm{Ric}(\nabla b^2, \nabla b^2) = 0 \ .$$

This in turn implies that M is flat $\mathbb{R}^3$.

□

Corollary 4.4.5 *Let (M^4, g) be a 4-dimensional Riemannian manifold. If $|\nabla b|^2$ is $\mathbf{C}^2$ function in a neighborhood of $x \in M$, then*

$$A'(r) = r \ Vol(\partial B_1(0)) \ \Delta |\nabla b|^2(x) + \frac{r}{2} \int\limits_{b \leq r} \left(\left|\mathrm{Hess}_{b^2} - \frac{\Delta b^2}{n} g\right|^2 + \mathrm{Ric}(\nabla b^2, \nabla b^2) \right) b^{-6} \ dVol \ .$$

Proof. The proof follows with similar arguments as in the case of 3-dimensional manifolds. However, in this case the second claim of Corollary 4.3.5 will be needed.

□

Chapter 5

New gradient estimates for Green's function

5.1 Sharp gradient estimates for the Green function

In this section, we will show how the monotonicity formulas we have studied so far are related to a sharp gradient estimate of the Green function. This will introduce later various results obtained from this sharp gradient estimate. As mentioned by Tobias Colding in his paper (see [5]), the correlation of our monotonicity formulas with this sharp gradient estimate for the Green function, which we will see below, is parallel to the fact that Perelman's monotonicity formula for the Ricci flow is closely related to the sharp gradient estimate of Li-Yau ([9]) for the heat kernel.

Theorem 5.1.1 *If (M^n, g) is any n-dimensional Riemannian manifold, with $n \geq 3$ and non-negative Ricci curvature, then for the function b, as defined above, it holds*

$$|\nabla b| \leq 1 \ .$$

Moreover, if equality holds at any point on M, then M is the flat Euclidean space $\mathbb{R}^n$.

Proof. From (ii) of Lemma 4.1.1, we have that

$$\lim_{r\to 0} \sup_{\partial B_r(x)} \left| |\nabla b|^2 - 1 \right| = 0 \ .$$

That is,

$$\forall \epsilon > 0, \ \exists \ r_0 \text{ such that } \sup_{\partial B_r(x)} \left| |\nabla b|^2 - 1 \right| < \epsilon \ , \ \forall r \leq r_0 \ .$$

We assume that $r > 0$ is sufficiently small, so that

$$\sup_{\partial B_r(x)} |\nabla b|^2 \leq 1 + \epsilon \ ,$$

where $B_r(x)$ a geodesic ball. Also we have from Lemma 4.3.2 that

$$|\nabla b| \le C = C(n) \ .$$

We consider the set $\{x : b(x) = R\}$, with R big enough and the smooth function $u : M \setminus \{x\} \to \mathbb{R}$, with

$$u = |\nabla b|^2 - (1+\epsilon) - C^2(n)\frac{b^{n-2}(x)}{R^{n-2}} \ .$$

Then, we have

$$\sup_{\partial B_r \cup \{x : b(x)=R\}} u \le 0 \ .$$

Moreover, since M has non-negative Ricci curvature, Lemma 4.3.2 again gives that

$$\mathcal{L}u \ge 0 \ .$$

By the maximum principle for the $\mathcal{L}$ operator[1] applied to u, we have for a fixed $y \in M \setminus \{x\}$ that

$$u \le 0 \ \text{ in the set } \{x : b(x) \le R\} \setminus B_r(x)$$

or, equivalently

$$|\nabla b|^2(y) \le 1 + \epsilon + C^2(n)\frac{b^{n-2}(y)}{R^{n-2}} \ .$$

Now letting $\epsilon \to 0$ and $R \to \infty$, the expected inequality follows; it gives a global gradient estimate for b on M, i.e. ,

$$|\nabla b| \le 1 \ .$$

We suppose that the function $|\nabla b|^2$ achieves its maximum value, i.e. ,

$$|\nabla b|^2(p) = 1 \ ,$$

at some $p \in M$. From the proof above, $|\nabla b|^2 \le 1$ and $\mathcal{L}|\nabla b|^2 \ge 0$. From these, it follows by the strong maximum principle this time, that

$$|\nabla b|^2 \equiv 1 \ ,$$

everywhere in the manifold, and thus from the first claim of Lemma 4.3.1, we get that

$$Hess_{b^2} = \frac{\Delta b^2}{n} g \qquad \text{and} \qquad \mathrm{Ric}(\nabla b^2, \nabla b^2) = 0 \ .$$

[1]Note that the maximum principles for the $\mathcal{L}$ operator are the same with those for the Laplacian one, since they differ in first-order derivatives.

Now, a paper of Cheeger-Colding ([2]) in chapter 1, implies that M is a metric cone and that the function b is the distance to the vertex. These, together with the fact that the euclidean space is the only smooth cone, we get the claim.

□

At this point we will give a second proof of the above theorem. To do so, we will use the maximum principle for the Laplace operator applied to the function u.

Alternative proof of the sharp bound for the gradient of b:
We have again by (ii) in Lemma 4.1.1 that for a given $\epsilon > 0$, we may choose $r > 0$ sufficiently small such that

$$\sup_{\partial B_r(x)} |\nabla b|^2 \leq 1 + \epsilon \ .$$

Also, by choosing $R > 0$ sufficiently large, together with the fact that for

$$r \to \infty \ \Rightarrow \ G \to 0 \ ,$$

we get

$$\sup_{\partial B_R} G \leq \epsilon \ .$$

Here, C is the gradient bound for b, given by Cheng-Yau gradient estimate. We consider a fuction u as follows

$$u = |\nabla b|^2 G - (1+\epsilon)G - C^2(n)\epsilon \ .$$

Then

$$\sup_{\partial B_r \cup \{x : b(x) = R\}} u \leq 0 \ .$$

Now, by Lemma 4.2.1 and since G is harmonic,

$$\Delta u = \Delta \left(|\nabla b|^2 G \right) \geq 0 \ ,$$

by Lemma 4.2.1. Thus, by the maximum principle for the Laplacian of u, the function u must be non-negative for each y in $M \setminus \{x\}$.

Hence we have at such a fixed y, that

$$u(y) \leq 0 \quad \Leftrightarrow \quad \left[|\nabla b|^2(y) - (1+\epsilon) \right] G(y) \leq C^2 \epsilon \ .$$

Letting $\epsilon \to 0$ and since $G(y)$ is non-negative, we have the inequality.

□

Remark. The argument in the proof of Theorem 5.1.1 in fact gives that

$$\sup_{b=r} |\nabla b|^2 \ ,$$

is a non-increasing function of r.

This observation is generalized in the following theorem.

Theorem 5.1.2 *Let Ω be an open bounded subset on M containing x. Then, for all $y \in M \setminus \Omega$,*

$$|\nabla b|^2(y) \leq \sup_{\partial\Omega} |\nabla b|^2 \ .$$

Moreover, strict inequality holds unless M is isometric to a cone outside a compact set.

Proof. The proof follows in the lines of the previous proof. We define

$$u = |\nabla b|^2(y) - (\sup_{\partial\Omega} |\nabla b|^2 + \epsilon)G - C^2\epsilon \ ,$$

and repeat all the arguments mentioned above.

□

Now we can write the sharp gradient estimate of Theorem 5.1.1 in terms of G.

Corollary 5.1.3 *If (M^n, g) is any n-dimensional non-parabolic Riemannian manifold, with $n \geq 3$ and non-negative Ricci curvature, then for the Green function G, we have*

$$|\nabla G| \leq (n-2)G^{\frac{n-1}{n-2}} \ .$$

Proof. We have directly from the inequality in Theorem 5.1.1, that

$$\Big|\frac{1}{G}\nabla G\Big| = |\nabla \log G| = (n-2)|\nabla \log b| = (n-2)\frac{|\nabla b|}{b} \leq (n-2)G^{\frac{1}{n-2}} \ .$$

Thus

$$|\nabla G| \leq (n-2)G^{\frac{n-1}{n-2}} \ .$$

□

Another immediate Corollary of the sharp gradient estimate for the Green function is the following, in which we will try to compare the volume of the sets which are defined via the level sets of the function b, with the volume of balls that we have in Euclidean space, respectively.

Corollary 5.1.4 *If M^n has non-negative Ricci curvature with $n \geq 3$, then, for all $r > 0$ we have*

$$\text{i)} \quad \mathrm{Vol}\left(\{x : b(x) = r\}\right) \geq \mathrm{Vol}(\partial B_r(0)) \ ,$$

$$\text{ii)} \quad \mathrm{Vol}\left(\{x : b(x) \leq r\}\right) \geq \int_0^r \mathrm{Vol}\left(\{x : b(x) = s\}\right) ds \geq \mathrm{Vol}(B_r(0)) \ .$$

Proof. By the sharp gradient estimate and the fact that $I_1(r)$ is constant as a fuction in r, we get

$$\mathrm{Vol}\left(\{x : b(x) = r\}\right) = \int_{b=r} 1 \ dArea$$

$$
\begin{aligned}
&\geq \int\limits_{b=r} |\nabla b| \; dArea \\
&= r^{n-1} I_1(r) \\
&= r^{n-1} \mathrm{Vol}(\partial B_1(0)) = \mathrm{Vol}(\partial B_r(0)) \; .
\end{aligned}
$$

For the second claim, by using the co area formula we have

$$
\begin{aligned}
\mathrm{Vol}\left(\{x : b(x) \leq r\}\right) &= \int\limits_{b \leq r} 1 \; d\mathrm{Vol} \\
&\geq \int\limits_{b \leq r} |\nabla b| \; d\mathrm{Vol} \\
&= \int\limits_0^r \int\limits_{b=s} \frac{|\nabla b|}{|\nabla b|} \; dArea \; ds \\
&= \int\limits_0^r \mathrm{Vol}\left(\{x : b(x) = s\}\right) ds \\
&\geq \int\limits_0^r \mathrm{Vol}(\partial B_s(0)) \; ds = \mathrm{Vol}(B_r(0)) \; .
\end{aligned}
$$

□

5.2 Sharp asymtotic gradient estimates for the Green function

In this section we show a sharp asymptotic gradient estimate for the Green function on manifolds with non-negative Ricci curvature.

Theorem 5.2.1 *Let (M^n, g) be an n-dimensional Riemannian manifold, with the assumption that $n \geq 3$. If M^n has non-negative Ricci curvature, then for the gradient of the function b we have*

$$
\lim_{r \to \infty} \sup_{M \setminus B_r(x)} |\nabla b| = \left[\frac{V_M}{\mathrm{Vol}(B_1(0))} \right]^{\frac{1}{n-2}} \; . \tag{5.2.1}
$$

To prove this theorem, we will need the following Lemma that was initially proved in [3] and was followed by a more complete description in Tobias Colding's paper.

Lemma 5.2.2 *Let M^n be an open manifold with non-negative Ricci curvature and let $u : M \to \mathbb{R}$ be a positive supeharmonic function on $B_r(x)$, $x \in M$. Then there exists a constant C, depending only on the dimension of M, such that*

$$\frac{1}{\mathrm{Vol}(B_r(x))} \int\limits_{\partial B_r(x)} u\ dArea \leq C\ u(x)\ .$$

Proof. Let h_r be the harmonic function on $B_r(x)$, with the property

$$h_r\big|_{\partial B_r(x)} = u\big|_{\partial B_r(x)}\ .$$

Then by the maximum principle we get that h_r is positive on $B_r(x)$, since u is positive on it.

Moreover, since u is also superharmonic on $B_r(x)$, we have that for h_r

$$h_r\big|_{B_r(x)} \leq u\big|_{B_r(x)}\ ,$$

and thus

$$0 < h_r(x) \leq u(x)\ .$$

Now, the Cheng-Yau Harnack inequality for h_r, applied on the ball $B_{\frac{r}{2}}(x)$, implies that for some constant $C = C(n)$, we have

$$\sup_{B_{\frac{r}{2}}(x)} h_r \leq C \inf_{B_{\frac{r}{2}}(x)} h_r \leq C\ u(x)\ .$$

From Stokes' theorem, and since h_r is harmonic and positive on $B_r(x)$, we have for some $s < r$ that

$$0 = \int_{B_s(x)} \Delta h_r\ d\mathrm{Vol} = \int_{\partial B_s(x)} \frac{\partial h_r}{\partial s}\ dArea\ .$$

By using standard properties of the Lie derivative and the Cartan's formula, we have

$$\begin{aligned}
\mathcal{L}_{\frac{\partial}{\partial s}}(h_r\ i_{\frac{\partial}{\partial s}}\ \omega_g) &= \mathcal{L}_{\frac{\partial}{\partial s}}(h_r)\ i_{\frac{\partial}{\partial s}}\ \omega_g + h_r\ \mathcal{L}_{\frac{\partial}{\partial s}}(i_{\frac{\partial}{\partial s}}\ \omega_g) \\
&= \frac{\partial}{\partial_s} h_r\ dS + h_r\ i_{\frac{\partial}{\partial s}}\ \mathfrak{L}_{\frac{\partial}{\partial s}}(\omega_g) \\
&= \frac{\partial}{\partial_s} h_r\ dS + h_r\ i_{\frac{\partial}{\partial s}}(\Delta s\ \omega_g) \\
&= \left(\frac{\partial}{\partial_s} h_r + h_r\ \Delta s\right) dS\ .
\end{aligned}$$

Hence, we get by a standard computation that

$$\frac{\partial}{\partial s}\left(\int_{\partial B_s(x)} h_r\ dArea\right) = \int\limits_{\partial B_s(x)} h_r\ \Delta s\ dArea + \int\limits_{\partial B_s(x)} \langle \nabla r, \nabla h_r \rangle dArea\ .$$

Then, from the Laplacian comparison theorem we get that

$$\int_{\partial B_s(x)} \frac{\partial h_r}{\partial s}\, dArea \;\geq\; \frac{\partial}{\partial s}\left(\int_{\partial B_s(x)} h_r\, dArea\right) - \frac{n-1}{s}\int_{\partial B_s(x)} h_r\, dArea\ ,$$

and equivalently,

$$\frac{\frac{\partial}{\partial s}\left(\int_{\partial B_s(x)} h_r\, dArea\right) - \frac{n-1}{s}\int_{\partial B_s(x)} h_r\, dArea}{\int_{\partial B_s(x)} h_r\, dArea} \leq 0\ . \tag{5.2.2}$$

For the left hand-side of the previous inequality, we have

$$\frac{(1-n)s^{-n}\int_{\partial B_s(x)} h_r\, dArea + s^{1-n}\frac{\partial}{\partial s}\left(\int_{\partial B_s(x)} h_r\, dArea\right)}{s^{1-n}\int_{\partial B_s(x)} h_r\, dArea} =$$
$$= \left[\log\left(s^{1-n}\int_{\partial B_s(x)} h_r\, dArea\right)\right]'\ .$$

Therefore (5.2.2) is equivalent to the fact that for fixed r, the function

$$f(s) = \log\left[s^{1-n}\int_{\partial B_s(x)} h_r\, dArea\right]\ ,$$

is non-increasing with respect to s.

All these impliy that,

$$\begin{aligned} r^{1-n}\int_{\partial B_r(x)} h_r\, dArea &\leq \left(\frac{r}{2}\right)^{1-n}\int_{\partial B_{\frac{r}{2}}(x)} h_r\, dArea \\ &\leq \left(\frac{r}{2}\right)^{1-n}\int_{\partial B_{\frac{r}{2}}(x)} 1\, dArea\ Cu(x) \\ &= C\left(\frac{r}{2}\right)^{1-n} Vol\left(\partial B_{\frac{r}{2}}(x)\right) u(x)\ . \end{aligned}$$

Finally, combining Bishop-Gromov inequality with the last computation we get the claim:

$$\begin{aligned} \frac{1}{\mathrm{Vol}(\partial B_r(x))}\int_{\partial B_r(x)} u\, dArea &= \frac{1}{\mathrm{Vol}(\partial B_r(x))}\int_{\partial B_r(x)} h_r\, dArea \\ &\leq 2^{n-1}C\frac{\mathrm{Vol}\left(\partial B_{\frac{r}{2}}(x)\right)}{\mathrm{Vol}\left(\partial B_r(x)\right)}u(x) \\ &\leq 2^n Cnu(x)\ . \end{aligned}$$

□

Now we are ready to see the Theorem's proof.

Proof. First we observe that in the case where M has sub-Euclidean volume growth, the claim follows trivially: Both sides of (5.2.1) are equal to zero. Note that the left hand side of (5.2.1) equals to zero, as this follows from the proof of Theorem 4.2.7

Now in the case where M has Euclidean volume growth, we consider as before the geodesic ball $B_r(x)$, centered at $x \in M$ and we set

$$L = \sup_{M \setminus B_r(x)} |\nabla b|^2 \ .$$

We also consider the ball $B_{2r}(x)$ and an arbitary point $y \in M \setminus B_{2r}(x)$. Since G is a positive harmonic function in the ball $B_{2r}(y)$, we get for some constant $C = C(n)$ that

$$\sup_{B_r(y)} G \leq C \inf_{B_r(y)} G \ ,$$

using Cheng-Yau Harnack inequality for the Green function.

Now for the function $G(L - |\nabla b|^2)$ we have

$$\Delta \left[G(L - |\nabla b|^2) \right] = -\Delta (G|\nabla b|^2) \leq 0 \ ,$$

with the last inequality following from Lemma 4.2.1, since the Ricci curvature is non-negative. This, using Lemma 5.2.2, applied to the positive and superharmonic function $G(L - |\nabla b|^2)$, on $B_r(y)$.

Hence we get

$$\begin{aligned} \frac{1}{C\mathrm{Vol}(\partial B_r(y))} \int\limits_{\partial B_r(y))} (L - |\nabla b|^2) \ dArea &\leq \frac{1}{\mathrm{Vol}(\partial B_r(y))} \int\limits_{\partial B_r(y))} G(L - |\nabla b|^2) \ dArea \\ &\leq C(L - |\nabla b|^2)(y) \ . \end{aligned}$$

We stress that the positive constant C depends only on the dimension of M (does not depend on r) [2].

Then, for the average of $L - |\nabla b|^2$ on the geodesic ball $B_r(y)$ we have

$$\frac{1}{\mathrm{Vol}(B_r(y))} \int\limits_{B_r(y)} (L - |\nabla b|^2) \ d\mathrm{Vol} \leq C(L - |\nabla b|^2)(y) \ ,$$

since

$$\begin{aligned} \frac{1}{\mathrm{Vol}(B_r(y))} \int\limits_{B_r(y))} (L - |\nabla b|^2) \ d\mathrm{Vol} &= \frac{1}{\mathrm{Vol}(B_r(y))} \int\limits_0^r \int\limits_{\partial B_s(y)} (L - |\nabla b|^2) \ dArea \ ds \\ &\leq \frac{1}{\mathrm{Vol}(B_r(y))} \ C(L - |\nabla b|^2)(y) \int\limits_0^r \mathrm{Vol}(\partial B_s(y)) \ ds \end{aligned}$$

[2] See for completeness the Corollary 6.1, with $\lambda = R = 0$, in [8].

$$= C(L - |\nabla b|^2)(y) \ .$$

Therefore, if we show that

$$\frac{1}{\mathrm{Vol}(B_r(y))} \int\limits_{B_r(y_r)} |\nabla b|^2 \ d\mathrm{Vol} \longrightarrow \left[\left(\frac{V_M}{\mathrm{Vol}(B_1(0))} \right)^{\frac{1}{n-2}} \right]^2 \ ,$$

as $r \to \infty$, with $y_r \in \partial B_{2r}(x)$, then we will have shown the claim. This is true, because if we choose a sequence (y_n) such that

$$|\nabla b|^2(y_n) \to L = \limsup_{y \to \infty} |\nabla b|^2(y) \ ,$$

then by setting $(r_n) = \frac{d(x,y_n)}{2}$, we get for $r \to \infty$ that

$$\begin{aligned} 0 \ &\leq \frac{1}{\mathrm{Vol}(B_{r_n}(y_n))} \int\limits_{B_{r_n}(y_n)} (L - |\nabla b|^2) \ d\mathrm{Vol} \\ &= L - \frac{1}{\mathrm{Vol}(B_{r_n}(y_n))} \int\limits_{B_{r_n}(y_n)} |\nabla b|^2 \ d\mathrm{Vol} \leq C(L - |\nabla b|^2)(y_n) \ , \end{aligned}$$

with the last tends to zero, as $n \to \infty$ and therefore we get

$$L = \left(\frac{V_M}{\mathrm{Vol}(B_1(0))} \right)^{\frac{2}{n-2}} \ .$$

So, we are left to show that the average of $|\nabla b|^2$ on all geodesic balls of radius r centered at $\partial B_{2r}(x)$ converges to the real and positive number $\left(\frac{V_M}{\mathrm{Vol}(B_1(0))} \right)^{\frac{2}{n-2}}$. In [7], (3.38) says that as r tends to infinity,

$$\int\limits_{b \leq r} \left| \lambda^2 - |\nabla b|^2 \right|^2 d\mathrm{Vol} = o \left[\mathrm{Vol}(\{b \leq r\}) \right] \ ,$$

where $\lambda = \left(\frac{V_M}{\mathrm{Vol}(B_1(0))} \right)^{\frac{1}{n-2}}$.

But the proof of the Theorem 4.2.7 shows that as $r \to \infty$,

$$\begin{aligned} \int\limits_{B_r(x)} \left| \lambda^2 - |\nabla b|^2 \right|^2 d\mathrm{Vol} &\leq \int\limits_{b \leq 2\lambda r} \left| \lambda^2 - |\nabla b|^2 \right|^2 d\mathrm{Vol} \\ &= o \left[\mathrm{Vol}(b \leq 2\lambda r) \right] \\ &= o \left[\mathrm{Vol}(B_r(x)) \right] \ , \end{aligned}$$

or,

$$\int\limits_{B_{3r}(x)} \left|\lambda^2 - |\nabla b|^2\right|^2 d\mathrm{Vol} = o\left[\mathrm{Vol}(B_{3r}(x))\right] \ .$$

But now, for the y_r defined above we have

$$B_r(y_r) \subseteq B_{3r}(x) \ , \tag{5.2.3}$$

and also

$$B_{3r}(x) \subseteq B_{5r}(y_r) \ .$$

By the Bishop-Gromov Volume Comparison theorem we immediately get that

$$\mathrm{Vol}(B_{3r}(x)) \leq \mathrm{Vol}(B_{5r}(y_r)) \leq 5^n \mathrm{Vol}(B_r(y_r)) \ . \tag{5.2.4}$$

Combining (5.2.3) and (5.2.4), we have

$$\int\limits_{B_r(y_r)} \left|\lambda^2 - |\nabla b|^2\right|^2 d\mathrm{Vol} \leq \int\limits_{B_{3r}(x)} \left|\lambda^2 - |\nabla b|^2\right|^2 d\mathrm{Vol} = o\left[\mathrm{Vol}(B_{3r}(x))\right] = o\left[\mathrm{Vol}(B_r(y_r))\right] \ .$$

Therefore, by the Holder inequality we get as $r \to \infty$ that

$$\begin{aligned}
\left|\lambda^2 \mathrm{Vol}(B_r(y_r)) - \int\limits_{B_r(y)} |\nabla b|^2 d\mathrm{Vol}\right| &\leq \int\limits_{B_r(y_r)} \left|\lambda^2 - |\nabla b|^2\right| d\mathrm{Vol} \\
&\leq \mathrm{Vol}(B_r(y_r))^{\frac{1}{2}} \left(\int\limits_{B_r(y_r)} \left|\lambda^2 - |\nabla b|^2\right|^2 d\mathrm{Vol} \right)^{\frac{1}{2}} \\
&= \mathrm{Vol}(B_r(y_r))^{\frac{1}{2}} o[\mathrm{Vol}(B_r(y_r))]^{\frac{1}{2}} \\
&= o[\mathrm{Vol}(B_r(y_r))] \ .
\end{aligned}$$

Then we have

$$\frac{1}{\mathrm{Vol}(B_r(y_r))} \int\limits_{B_r(y_r)} |\nabla b|^2 d\mathrm{Vol} \to \lambda^2 = \left(\frac{V_M}{\mathrm{Vol}(B_1(0))}\right)^{\frac{2}{n-2}}$$

as claimed.

□

As before, in the sharp gradient estimate, we can write this as an asymptotic gradient estimate for the Green function, which is in a sense, one of the most important of our goals, in this thesis.

Corollary 5.2.3 *If (M^n, g) is any n-dimensional non-parabolic Riemannian manifold, with $n \geq 3$, then for the minimal positive Green function G, we have*

$$\frac{1}{n-2} \lim_{r \to \infty} \sup_{M \setminus B_r(x)} \frac{|\nabla G|}{G^{\frac{n-1}{n-2}}} = \left[\frac{V_M}{\mathrm{Vol}(B_1(0))} \right]^{\frac{1}{n-2}} .$$

Proof. The claim follows directly from the last Theorem, since

$$|\nabla b| = \frac{1}{n-2} \, \frac{|\nabla G|}{G^{\frac{n-1}{n-2}}} .$$

□

Chapter 6

Appendix

6.1 Bochner-Weitzenbock formula

As it is proved in ([14]), we have the following theorem, which is known as Bochner-Weitzenbock formula.

Theorem 6.1.1 *Let (M^n, g) be a complete Riemannian manifold. Then for any function $f \in C^2(M)$, we have*

$$\frac{1}{2}\Delta|\nabla f|^2 = |\mathrm{Hess} f|^2 + \langle \nabla f, \nabla(\Delta f)\rangle + \ \mathrm{Ric}(\nabla f, \nabla f)\ ,$$

pointwise.

Proof: For a fixed point $p \in M$, let $\{X_1, ..., X_n\}$ be a local orthonormal frame, such that

$$\langle X_i, X_j\rangle = \delta_{ij}, \quad \nabla_{X_i} X_j(p) = 0\ .$$

In terms of a local coordinates system $\{x^1, ..., x^n\}$ at p, the Laplace Beltrami operator is defined by

$$\Delta f = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}\ g^{ij}\frac{\partial f}{\partial x^i}\right)\ .$$

Using the fact that Hessian is symmetric, the classical Leibniz rule and properties of the curvature tensor, we have

$$\frac{1}{2}\Delta|\nabla f|^2 = \Delta(\langle \nabla f, \nabla f\rangle) = \frac{1}{2}\sum_{i=1}^{n} X_i X_i \langle \nabla f, \nabla f\rangle =$$

$$= \frac{1}{2}\sum_{i=1}^{n} X_i \left(\langle \nabla_{X_i}\nabla f, \nabla f\rangle + \langle \nabla f, \nabla_{X_i}\nabla f\rangle\right) =$$

$$= \sum_{i=1}^{n} X_i \langle \nabla_{X_i}\nabla f, \nabla f\rangle = \sum_{i=1}^{n} X_i \cdot \mathrm{Hess} f(X_i, \nabla f) =$$

$$= \sum_{i=1}^{n} X_i \cdot \mathrm{Hess}f(\nabla f, X_i) = \sum_{i=1}^{n} X_i \langle \nabla_{\nabla f}(\nabla f), X_i \rangle =$$

$$= \sum_{i=1}^{n} \langle \nabla_{X_i} \nabla_{\nabla f}(\nabla f), X_i \rangle + \langle \nabla_{\nabla f}(\nabla f), \nabla_{X_i} X_i \rangle =$$

$$= \sum_{i=1}^{n} \langle R(X_i, \nabla f)\nabla f, X_i \rangle + \sum_{i=1}^{n} \langle \nabla_{\nabla f} \nabla_{X_i} \nabla f, X_i \rangle +$$

$$+ \sum_{i=1}^{n} \langle \nabla_{[X_i, \nabla f]} \nabla f, X_i \rangle \ . \quad (6.1.1)$$

Here, the first term is by definition equal to $\mathrm{Ric}(\nabla f, \nabla f)$.

The second term of (1.1) is equal to,

$$\sum_{i=1}^{n} (\nabla f \langle \nabla_{X_i} \nabla f, X_i \rangle - \langle \nabla_{X_i} \nabla f, \nabla_{\nabla f} X_i \rangle) =$$

$$\sum_{i=1}^{n} \nabla f \langle \nabla_{X_i} \nabla f, X_i \rangle = \nabla f \sum_{i=1}^{n} \langle \nabla_{X_i} \nabla f, X_i \rangle =$$

$$\nabla f \sum_{i=1}^{n} \mathrm{Hess}f(X_i, X_i) = \nabla f(\Delta f) = \langle \nabla f, \nabla(\Delta f) \rangle \ . \quad (6.1.2)$$

Finally, for the last term of (1.1) we have

$$\sum_{i=1}^{n} \langle \nabla_{[X_i, \nabla f]} \nabla f, X_i \rangle = \sum_{i=1}^{n} \mathrm{Hess}f([X_i, \nabla f], X_i) =$$

$$\sum_{i=1}^{n} \mathrm{Hess}f(\nabla_{X_i} \nabla f - \nabla_{\nabla f} X_i, X_i) = \sum_{i=1}^{n} \mathrm{Hess}f(X_i, (\nabla_{X_i} \nabla f) =$$

$$\sum_{i=1}^{n} \langle \nabla_{X_i} \nabla f, \nabla_{X_i} \nabla f \rangle = \sum_{i=1}^{n} |\nabla_{X_i} \nabla f|^2 = \sum_{i=1}^{n} \nabla_{X_i} \nabla_{X_i} |\nabla f|^2 =$$

$$|\mathrm{Hess}f|^2 \ . \quad (6.1.3)$$

Combining (1.2),(1.3) and (1.4) the claim follows. Note that for $\mathbb{R}^n$, the Bochner-Weitzenbock formula becomes

$$\frac{1}{2} \Delta |\nabla f|^2 = |\mathrm{Hess}f|^2 + \langle \nabla f, \nabla(\Delta f) \rangle \ ,$$

namely missing the curvature term.

6.2 The Co-area formula on Riemannian manifolds

Co-area formula extends naturally from the Euclidean space to Riemannian manifolds as follows:

Theorem 6.2.1 *Let (M, g) be a Riemannian manifold and $f \in C_c^\infty(M)$ a fixed function. Then, for each $h \in L^1(M)$ we have*

$$\int_M h d\mu_g = \int_{\mathbb{R}} \int_{\Gamma(t)} \frac{h}{|\nabla f|} dA(t) dt \ , \tag{6.2.1}$$

where $\Gamma(t) := f^{-1}(\{t\}) = \{p \in M : f(p) = t\}$ and $dA(t)$ is the induced measure on $\Gamma(t)$.

Proof. By Sard's theorem, the critical values of f have measure zero, so $\Gamma(t)$ is an immersed $(n-1)-$dimensional submanifold of M for almost all $t \in f(M) \subseteq \mathbb{R}$. We also have that $\mathrm{Reg}(f)$, the set of all regular values of f, is an open subset of $\mathbb{R}$. Let $(\alpha, \beta) \subseteq \mathrm{Reg}(f)$ and pick any fixed number $c \in (\alpha, \beta)$. Let φ_{t-c} be the flow determined by the vector field $\frac{\nabla f}{|\nabla f|^2}$ restricted to $f^{-1}\big((a, b)\big)$, that is

$$\frac{d}{dt}\varphi_{t-c}(p) = \frac{\nabla f}{|\nabla f|^2} \text{ and } \varphi_{t-c}(p)\Big|_{t=c} = p \ , \text{ for any } p \in f^{-1}\big((\alpha, \beta)\big) \ .$$

Clearly, the mapping

$$\Phi : f^{-1}(c) \times (\alpha, \beta) \longrightarrow f^{-1}\big((\alpha, \beta)\big) \ , \ (p, t) \longmapsto \Phi(p, t) := \varphi_{t-c}(p) \ ,$$

defines a diffeomorphism onto $f^{-1}\big((\alpha, \beta)\big)$. For all $(p, t) \in f^{-1}(c) \times (\alpha, \beta)$ we have

$$f\big(\Phi(p, c)\big) = f(\varphi_0(p)) = f(p) = c,$$
$$\frac{d}{dt} f\big(\Phi(p, c)\big) = \nabla f\Big|_{\Phi(p,c)} \cdot \frac{d}{dt}\big(\varphi_{t-c}(p)\big) = \Big(\nabla f \cdot \frac{\nabla f}{|\nabla f|^2}\Big)\Big|_{\Phi(p,c)} = 1 \ .$$

Thus, for any $(p, t) \in f^{-1}(c) \times (\alpha, \beta)$,

$$f(\Phi(p, t)) = t \Longrightarrow \Phi(p, t) \in \Gamma(t) \ .$$

Furthermore, $\frac{\partial \Phi(t)}{\partial t} \perp \Gamma(t)$ since $\frac{\partial \Phi(t)}{\partial t} = \frac{\nabla f}{|\nabla f|^2}$ is parallel to ∇f and ∇f is perpendicular to $\Gamma(t)$. Since Φ is diffeomorphism we can choose local coordinates (p, t) on the open $f^{-1}\big((\alpha, \beta)\big) \subseteq M$ where $p \in \Gamma(t)$ and $t \in (\alpha, \beta)$. Specifically, if $G_{ij}(t)$ $(i, j = 1, 2, \ldots, n-1)$ are the components of the metric on $\Gamma(t)$, we may write

$$g\Big|_{f^{-1}\big((\alpha,\beta)\big)} = \begin{pmatrix} \text{metric on } \Gamma(t) & 0 \\ 0 & \text{metric on } \big(\Gamma(t)\big)^\perp \end{pmatrix} = \begin{pmatrix} G_{ij}(t) & 0 \\ 0 & |\frac{\partial \Phi}{\partial t}|^2 \end{pmatrix} \ .$$

Thus, locally we have

$$\begin{aligned}
d\mu_g &= \sqrt{det(g_{ij})}dx_1 \wedge \dots dx_{n-1} \wedge dt \\
&= \sqrt{|\frac{\partial \Phi}{\partial t}|^2 det(G_{ij}(t))} dx_1 \wedge \dots dx_{n-1} \wedge dt \\
&= |\frac{\partial \Phi}{\partial t}| \sqrt{\det(G_{ij}(t))} dx_1 \wedge \dots dx_{n-1} \wedge dt \\
&= \Big|\frac{\nabla f}{|\nabla f|^2}\Big| \Big(\sqrt{\det(G_{ij}(t))} dx_1 \wedge \dots dx_{n-1}\Big) \wedge dt \\
&= \frac{1}{|\nabla f|} dA(t) \wedge dt \ .
\end{aligned}$$

and therefore, for any function $h \in L^1(M)$ we have

$$hd\mu_g = \frac{h}{|\nabla f|} dA(t) \wedge dt \Longrightarrow \int_M h d\mu_g = \int_{\mathbb{R}} \int_{\Gamma(t)} \frac{h}{|\nabla f|} dA(t) \wedge dt \ .$$

□

6.3 Laplacian Comparison Theorem

The Laplacian Comparison Theorem[1] is a fundamental result in Riemannian geometry. It is directly related to the volume comparison theorem and also a special case of the Rauch comparison theorem.

Theorem 6.3.1 *Let (M^n, g) be a complete Riemannian manifold with Ricci curvature bounded from below by*

$$\mathrm{Ric} \geq (n-1)k \ ,$$

for some constant k. Suppose $p \in M$ a fixed point and we consider the distance function $r(x)$ be smooth. Then for any $x \in M$, the Laplacian of the distance function satisfies

$$\Delta r = \begin{cases} (n-1)\sqrt{k}\cot\sqrt{k}r & \text{for } k > 0 \ , \\ \frac{n-1}{r} & \text{for } k = 0 \ , \\ (n-1)\sqrt{-k}\cot\sqrt{-k}r & \text{for } k < 0 \ . \end{cases}$$

Proof. From the Bochner-Weitzenbock formula for any function $f \in C^2(M)$ we have

$$\frac{1}{2}\Delta|\nabla f|^2 = |\mathrm{Hess} f|^2 + \langle \nabla f, \nabla(\Delta f)\rangle + \ \mathrm{Ric}(\nabla f, \nabla f) \ .$$

[1] For more details we refer to see [14].

Now, letting $f(x) = r(x)$ we get, outside the cut locus of p, that

$$0 = |\text{Hess}r|^2 + \frac{\partial}{\partial r}(\Delta r) + \text{Ric}(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) \ ,$$

since $|\nabla r| = 1$.

We assume $\lambda_1, ..., \lambda_n$ be the eigenvalues of Hess$r$, and hence we have that $\lambda_i = 0$, with $i \in \{1, ..., n\}$, since the exponential function is a radial isometry.

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |\text{Hess}r|^2 &= \lambda_1^2 + ... + \lambda_n^2 \\ &\geq \frac{(\lambda_1 + ... + \lambda_n)^2}{n-1} \\ &= \frac{\big(\text{tr}(\text{Hess}r)\big)^2}{n-1} \\ &= \frac{(\Delta r)^2}{n-1} \ . \end{aligned}$$

Since we consider Ric $\geq (n-1)k$, the last inequality together with the Bochner-Weitzenbock identity, imply

$$\frac{(\Delta r)^2}{n-1} + \frac{\partial}{\partial r}(\Delta r) + (n-1)k \leq 0 \ . \tag{6.3.1}$$

If we let now $u = \frac{n-1}{\Delta r}$, we get from (6.3.1) that

$$\begin{aligned} &(n-1)\frac{\frac{\partial}{\partial r}(\Delta r)}{(\Delta r)^2} + \frac{(n-1)^2}{(\Delta r)^2}k + 1 \leq 0 \\ \Leftrightarrow \ &\frac{u'}{u^2 k + 1} \geq 1 \ . \end{aligned}$$

Note that $\Delta r \to \frac{n-1}{r}$ when $r \to 0$; thus $u \to r$. Now, integrating the above inequality, we get the claim.

□

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